q -Calculus and the Symmetrized Radioactive Decay Law

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Abstract: The radioactive decay law was first formulated by Ernest Rutherford and Frederick Soddy in 1902. As a well-known law, one of its primary applications is to determine the dates of ancient specimens. The process is known as radiocarbon dating and is subjected to the known properties of radioactive nuclei. In this paper, we implement quantum calculus to express the solution of the radioactive decay equation in symmetrized q -exponential form. Also, we explore a q -analog of the decay constant using Tsallis logarithmic function for various miscellaneous q -values. Furthermore, the factor-label method was applied to our analysis to show that the correct units remained intact under the application of quantum calculus. In conclusion, our work suggests that a variation of the q -parameter was akin to the production of a new isotope for all q in $(0,1)$; the superadditive regime.

Keywords: q -calculus, radioactive decay, half-life, factor-label method

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1 Introduction

In this paper, we will show by means of the power series method and quantum calculus in Section (3), that the radioactive decay law can be written in symmetrized quantum exponential form. In addition to that, we will use Tsallis nonextensive statistical mechanics to generate a number of superadditive curves. Superadditive curves are defined as having a q -parameter that is less than unity, i.e., in our case $q \in (0, 1)$.

To generate the superadditive curves, we intend to exploit a special property of the q -logarithm based on Tsallis statistics. Our results will then be applied to the radioactive decay constant. Finally, in Section (4) we will use the factor-label method to show that the units remained intact under a variation of the q -parameter.

1.1 Classical decay

A known solution to the radioactive decay equation can be yielded from an elementary technique in ordinary differential equations known as the separation of variables[1]. Of course, this is not the only method to solve such an equation. Another method which ensures an identical solution to the previous equation is the power series method.

However, if we let the initial conditions of our system be as follows: at time $t = 0$, $N(t) = N_o = a_o$. Then the solution by the power series method is determined by substituting Eq.(1.2) in Eq.(1.1) to give Eq.(1.3),

$$\frac{dN}{dt} = -\lambda N \quad (1.1)$$

$$N(t) = \sum_{n=0}^{n=\infty} a_n t^n \quad (1.2)$$

$$\sum_{n=0}^{n=\infty} (n+1) a_{n+1} t^n + \lambda \sum_{n=0}^{n=\infty} a_n t^n = 0. \quad (1.3)$$

The terms a_n and a_{n+1} are related and hence the power series solution in ref.[2] is attained by substituting a_n in Eq.(1.2) where,

$$a_n = \frac{(-1)^n (\lambda)^n}{n!} a_0 \quad (1.4)$$

and

$$N(t) = \sum_{n=0}^{n=\infty} \frac{(-1)^n (\lambda)^n}{n!} a_0. \quad (1.5)$$

A quicker and perhaps more straightforward method to find the solution to Eq.(1.1) is by separating variables[3]. The terms N_0 , $N(t)$, λ , and t are: (i) the number of nuclei at $t = 0$, (ii) the number of nuclei at $t > 0$, (iii) the radioactive decay constant, and (iv) the time t , respectively. Additionally, if we integrate Eq.(1.1) using the method of separation of variables, we get

$$\int_{N_0}^{N(t)} \frac{dN}{N} = -\lambda \int_0^t dt \quad (1.6)$$

$$\boxed{N(t) = N_o \exp(-\lambda t).} \quad (1.7)$$

1.2 Quantum calculus

Quantum calculus in ref.[4] or sometimes called q -Calculus is calculus done without limits. The term h in $q = e^h$ is used as a reminder of the planckian constant, while the term q stands for "quantum". In the limit $h \rightarrow 0$, $q \rightarrow 1$ and the classical form of calculus is recovered[5]. Furthermore, another important concept in q -calculus which will be discussed later in more detail, is the derivative and its q -analog—it is important to point out that currently no q -analog for the generalized chain rule exist [5]. We will give an example of a function which satisfies the chain rule in Section (2), but bear in mind that several functions exist that do not satisfy this rule.

Nonetheless, for the sake of clarity the downward arrow between Eqs.(1.8) and (1.9) illustrates the removal of the limit. The limit in Eq.(1.8) is replaced by the number "1" and

hence we get the well-known *Jackson derivative* in ref.[6–9] which is given by Eq.(1.9)

$$\frac{df}{dx} = \lim_{q \rightarrow 1} \frac{f(qx) - f(x)}{qx - x} \quad (1.8)$$

$$\downarrow$$

$$\frac{d_q f}{d_q x} = \left\{ 1 \right\} \frac{f(qx) - f(x)}{qx - x}. \quad (1.9)$$

2 q -Basics

2.1 The q -integer

Definition 2.1. Given that $0 < q < 1$ and $n \in \mathbb{N}$, the q -integer is defined by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & \text{if } q \neq 1, \\ n, & \text{if } q = 1, \end{cases} \quad (2.1)$$

for $q \in \mathbb{R}$.

The next definition we will consider is the q -factorial in [10].

2.2 The q -factorial

Definition 2.2. Let $0 < q < 1$ and $n \in \mathbb{N}$, then the q -factorial is defined as

$$[n]_q! = \begin{cases} [n]_q [n-1]_q \dots [1]_q, & n = 1, 2, 3, \dots, \\ 1, & n = 0. \end{cases} \quad (2.2)$$

2.3 Tsallis q -logarithm

Definition 2.3. Let q be a real parameter defined on the open interval $(0, 1)$, then the Tsallis q -logarithm is defined as

$$\ln_q(x) = \begin{cases} \ln(x), & x > 0, \text{ and } q = 1, \\ \frac{x^{1-q} - 1}{1-q}, & x > 0, \text{ and } q \neq 1, \\ \text{undefined}, & \text{if } x \leq 0. \end{cases} \quad (2.3)$$

Therefore, using Def.(2.3), we conveniently show that in the limit as $q \rightarrow 1$, $\ln_q(x)$ goes to

$$\lim_{q \rightarrow 1} \{\ln_q(x)\} = (-1) \lim_{q \rightarrow 1} \frac{d}{dq} (x^{1-q}), \quad (2.4)$$

and

$$\lim_{q \rightarrow 1} \{\ln_q(x)\} = \lim_{q \rightarrow 1} \frac{d}{du} (e^{u \ln(x)}) = \ln(x), \quad (2.5)$$

where we have made the substitution ($u = 1 - q$), in Eq.(2.4).

Given Def.(2.3), our analysis of the radioactive decay law requires the use of Tsallis nonextensive statistical mechanics and probability theory. Therefore, we will give a brief description of how we will use the theory to justify our results. Now given two independent systems Q^α and Q^γ ; thereby, from probability theory we know that $p(Q^\alpha + Q^\gamma) = p(Q^\alpha)p(Q^\gamma)$. It follows then that—Tsallis entropy— $S_q(Q^\alpha + Q^\gamma) = S_q(Q^\alpha) + S_q(Q^\gamma)$, which is the limiting case—representing additivity when $q = 1$ —and satisfies Boltzmann-Gibbs statistics.

For non-integer representations, i.e., $q \in (0, 1)$ we get the well-known Tsallis nonextensive statistical mechanics—extremization—and hence $S_q(Q^\alpha + Q^\gamma) = S_q(Q^\alpha) + S_q(Q^\gamma) - (q - 1)S_q(Q^\alpha)S_q(Q^\gamma)$ in ref.[11]. Thus whenever the entropic parameter $q < 1$ —superadditivity—it must be true that $S_q(Q^\alpha + Q^\gamma) \geq S_q(Q^\alpha) + S_q(Q^\gamma)$.

2.4 The q -derivative

Definition 2.4. Let $f(x)$ be a function that is differentiable and hence continuous; then its q -derivative is define as

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0, \quad (2.6)$$

where q in quantum calculus is described as the so-called q -parameter and varies strictly between 0 and 1.

Clearly, from Def.(2.4), in the limit that $q \rightarrow 1$,

$$\lim_{q \rightarrow 1} (D_q f)(x) = \frac{df(x)}{dx} \quad (2.7)$$

and we recover the classical form of the derivative. Likewise, following from the definition of the q -derivative[12], for constants α and β there exist a linear relationship such that

$$D_q\{\alpha f(x) + \beta g(x)\} = \alpha D_q\{f(x)\} + \beta D_q\{g(x)\}. \quad (2.8)$$

The next formula we shall consider is the product rule. In traditional—classical—calculus, if we take into consideration the product of two functions, $f(x)g(x)$ —the derivative can be applied to the first function while holding the second function, followed by an addition sign and then the derivative is applied to second function while holding the first function [13]. Therefore, we will show that the product rule in q -calculus is nearly identical to its counterpart in classical calculus. The only difference between them is the q -parameter appears in the argument of the first function, i.e., $f(x)$ becomes $f(qx)$ and the q -derivative operates on each individual function as it would in classical calculus. Henceforth, using Eqs.(2.6) and (2.8), we may write the product formula for $x \neq 0$ as

$$\begin{aligned} D_q\{f(x)g(x)\} &= \frac{f(qx)g(qx) - f(x)g(x)}{(q-1)x} \\ &= \frac{f(qx)g(qx) - f(qx)g(x)}{(q-1)x} \\ &\quad + \frac{f(qx)g(x) - f(x)g(x)}{(q-1)x} \\ &= \frac{f(qx)(g(qx) - g(x))}{(q-1)x} \\ &\quad + \frac{(f(qx) - f(x))g(x)}{(q-1)x} \\ &= f(qx)D_qg(x) \\ &\quad + D_qf(x)g(x). \end{aligned} \quad (2.9)$$

If we consider an interchange of the functions $f(x)$ and $g(x)$, then

$$\begin{aligned} D_q\{f(x)g(x)\} &= f(x)D_qg(x) + g(qx)D_qf(x). \end{aligned} \quad (2.10)$$

It follows then that Eqs.(2.9 and 2.10) are equivalent forms. We say these forms are symmetric since an interchange of any of the two functions does not change the result. The symmetric nature of quantum calculus is important to some physicist because of its application to algebraic objects called quantum groups¹ in [10]. A quantum group can be viewed as a Hopf algebra, i.e., a bialgebra—an algebra and a co-algebra—with an antipode. That is, for a Lie algebra S :

- $S \times S \rightarrow S$ (multiplicative map),
- $S \rightarrow S : g \rightarrow g^{-1}$ (inversion map),
- $\Delta : S \rightarrow S \otimes S$ (comultiplication),
- $\delta : S \rightarrow S$ (antipode), and
- $\zeta : S \rightarrow \mathbb{R}$ (counit).

In like manner, the quotient rule and a non-generalized example that leads to a limited definition of the chain rule—no generalized chain rule currently exist—are given by[5],

$$\begin{aligned} D_q\left\{\frac{f(x)}{g(x)}\right\} &= \frac{1}{(q-1)x} \left\{ \frac{f(qx)}{g(qx)} - \frac{f(x)}{g(qx)} \right\} \\ &\quad + \frac{1}{(q-1)x} \left\{ \frac{f(x)}{g(qx)} - \frac{f(x)}{g(x)} \right\} \\ &= \frac{1}{g(qx)} \left\{ \frac{f(qx) - f(x)}{(q-1)x} \right\} \\ &\quad + \frac{1}{(q-1)x} \left\{ \frac{f(x)g(x) - f(x)g(qx)}{g(qx)g(x)} \right\} \\ &= \frac{1}{g(qx)} D_qf(x) \\ &\quad + \frac{f(x)}{g(qx)g(x)} \left\{ \frac{g(x) - g(qx)}{(q-1)x} \right\} \\ &= \frac{g(x)D_qf(x) - f(x)D_qg(x)}{g(qx)g(x)}. \end{aligned} \quad (2.11)$$

¹Quantum groups are algebraic objects that are needed in the description of the 2+1 gravity.

And for some arbitrary function of the form $f(u(x)) = f(\alpha x^\beta)$ in [5], where α and β are constants, we get the following equation for a special case of the chain rule as depicted in ref.[14],

$$\begin{aligned} D_q\{f(u(x))\} &= \\ D_q\{f(\alpha x^\beta)\} &= \frac{f(\alpha q^\beta x^\beta) - f(\alpha x^\beta)}{(q-1)x} \\ &= \frac{f(\alpha q^\beta x^\beta) - f(\alpha x^\beta)}{\alpha q^\beta x^\beta - \alpha x^\beta} \times \frac{\alpha q^\beta x^\beta - \alpha x^\beta}{(q-1)x} \\ &= (D_{q^\beta} f)(u(x)) D_q(u(x)). \end{aligned} \quad (2.12)$$

2.5 The q -exponential

Definition 2.5. The q -exponential function (with $q \in (0, 1)$) is defined by

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}. \quad (2.13)$$

It should be obvious—from Def.(2.2)—that the classical exponential function (e^x) is recovered in the limit as $q \rightarrow 1$. The term in the denominator of Eq.(2.13); $[n]_q! = [1]_q[2]_q \dots [n]_q$ in [15] is the only major difference between e^x and $e_q(x)$ and this is where the symmetry and invariance arises from. If $q \rightarrow \frac{1}{q}$ and $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, then $e_q(x) = e_{\frac{1}{q}}(x)$ are equivalent. Again, the fact that in the limiting case one can recover the classical form makes quantum calculus exceptionally beautiful[16].

3 q -Exponential form

3.1 The symmetrized form

The precursory definition of the q -exponential function in Subsection (2.5) will enable us to write the solution for the radioactive decay equation by means of a suitable power series[17]. Conversely a q -exponential function based on Tsallis nonextensive statistical mechanics will be written in terms of Def.(2.3). That is, λ will be written in terms of Tsallis q -logarithm. It will then be shown graphically—if we let the q -parameter vary for values strictly between 0 and 1 [18], that

the classical form is indeed recovered whenever $q \rightarrow 1$.

We shall accomplish this by choosing a known isotope of oxygen—oxygen-22—in [19] with a corresponding *half-life* of 2.25 seconds. The choice of isotope is completely random, and there exist no reason why any other isotope with a known *half-life* would not serve the same purpose. The modified decay constant is the given by

$$\lambda_q = \frac{1}{2.25s} \times \ln_q(2). \quad (3.1)$$

That being said, let the power series for $N(t)$ be defined as depicted in Eq.(1.2); then

$$\frac{d_q N(t)}{d_q t} = \sum_{n=0}^{\infty} [n]_q a_n t^{n-1}. \quad (3.2)$$

Now given that $[0]_q = 0$, then

$$\frac{d_q N(t)}{d_q t} = [0]_q a_0 t^{-1} + \sum_{n=1}^{\infty} [n]_q a_n t^{n-1} \quad (3.3)$$

for $n \geq 1$. We then substitute Eq.(3.3) in Eq.(1.1) and this gives,

$$\sum_{n=0}^{\infty} [n+1]_q a_{n+1} t^n + \lambda \sum_{n=0}^{\infty} a_n t^n = 0. \quad (3.4)$$

The terms in Eq.(3.4), which are of the form a_n and a_{n+1} [20] are the coefficients of t^n , therefore we must equate them, which gives

$$a_{n+1} = -\frac{\lambda a_n}{[n+1]_q}. \quad (3.5)$$

The relationship between a_n and a_{n+1} allows us to generate the following terms:

$$a_1 = -\frac{\lambda a_0}{[1]_q}, \quad (3.6)$$

$$a_2 = \frac{(-\lambda)(-\lambda)a_0}{[1]_q[2]_q}, \quad (3.7)$$

$$a_3 = \frac{(-\lambda)(-\lambda)(-\lambda)a_0}{[1]_q[2]_q[3]_q}, \quad (3.8)$$

$$a_4 = \frac{(-\lambda)(-\lambda)(-\lambda)(-\lambda)a_0}{[1]_q[2]_q[3]_q[4]_q}, \quad (3.9)$$

$$a_5 = \frac{(-\lambda)(-\lambda)(-\lambda)(-\lambda)(-\lambda)a_0}{[1]_q[2]_q[3]_q[4]_q[5]_q}, \text{ and} \quad (3.10)$$

$$a_6 = \frac{(-\lambda)(-\lambda)(-\lambda)(-\lambda)(-\lambda)(-\lambda)a_0}{[1]_q[2]_q[3]_q[4]_q[5]_q[6]_q}. \quad (3.11)$$

Given that,

$$[1]_q[2]_q[3]_q \dots [n]_q = [n]_q! \quad (3.12)$$

we may write a_n as,

$$a_n = \frac{(-1)^n \lambda^n a_0}{[n]_q!}. \quad (3.13)$$

We may now substitute a_n in Eq.(1.2), giving us

$$N_q(t) = N_0 \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n t^n}{[n]_q!}, \quad (3.14)$$

and

$$N_{\frac{1}{q}}(t) = N_q(t) = N_0 e_q(-\lambda t) = N_0 e_{\frac{1}{q}}(-\lambda t) \quad (3.15)$$

which is the q -analog of the symmetrized radioactive decay law. As a result of $e_q(x) = e_{\frac{1}{q}}(x)$, $N_{\frac{1}{q}}(t) = N_q(t)$ must be symmetric whenever $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$.

3.2 The Half-Life

The *half-life* ($t_{1/2}$) is the time it takes for the number of radioactive nuclei to decrease by half the original value [21]. In the classical form, it is written as

$$t_{1/2} = \frac{\ln(2)}{\lambda} \quad (3.16)$$

where λ is the radioactive constant. The fact that $\ln_q(x)$ exist for $x > 0$ and $q \neq 1$, implies we may write $t_{1/2}$ as $t_{(1/2)Oxy22} = \frac{\ln_q(2)}{\lambda_q}$, where $t_{(1/2)Oxy22}$ is the *half-life* of oxygen-22. Again, we can move in the reverse direction to recover the classical

form when $q \rightarrow 1$ as depicted below in Eq.(3.17),

$$\begin{aligned} t_{1/2} &= \lim_{q \rightarrow 1} \{t_{(1/2)Oxy22}\} \\ &= -\lambda^{-1} \lim_{q \rightarrow 1} \frac{d}{dq} \left\{ 2^{1-q} \right\} \\ &= \frac{\ln(2)}{\lambda}. \end{aligned} \quad (3.17)$$

In addition to that, we will complete our plots with an equation of the form $\tilde{e}_q^{\ln_q(x)}$, where tilde differentiates $\tilde{e}_q^{\ln_q(x)}$ from Def.(2.5)—thus we take advantage of a property of Tsallis functions, which is related to nonextensive statistical mechanics²—for several miscellaneous q -values. Tsallis functions are credited to Constantin Tsallis—1988—and have a variety of applications not only in physics but also in engineering and applied mathematics. Note that in [22],

$$\ln_q e_q^x = e_q^{\ln_q x} = x, \quad \forall x; \quad (3.18)$$

where

$$e_q^x \equiv [1 + (1 - q)x]^{\frac{1}{1-q}}. \quad (3.19)$$

Remark 1. A function is said to be strictly increasing on an interval I , if $\forall p, q \in I$, $f(q) > f(p)$. It turns out that the q -logarithm is a strictly increasing function. The action of taking the first derivative with respect to x , given Def.(2.3), when $q \neq 1$ and $x > 0$, gives

$$\frac{d}{dx} \ln_q(x) = \frac{1}{x^q}. \quad (3.20)$$

This result holds true $\forall q$.

3.3 q -Exponential plots

In order to plot the superadditive curves, the miscellaneous q -values in Table (1) were attained from the open interval $(0, 1)$; that is, $q \in (0, 1)$. We selected q -values of: 0.1, 0.2, 0.3, 0.7, and 0.9 respectively. These values were substituted in the Tsallis q -logarithm— λ_q —and the numerical answers were divided by the *half-life* of oxygen-22 which is approximately 2.25 seconds.

² Tsallis nonextensive statistical mechanics is a generalization of Boltzmann-Gibbs statistical mechanics.

The final result is substituted into $\gamma_0 N_q(t)$ and plotted for values of $t = [0, 20]$, since $t_{(1/2)Oxy22} = \frac{\ln_q(2)}{\lambda_q}$. This will permit us to visualize the behavior of a curve; for some $q \in (0, 1)$. Nevertheless, in the limiting case (when $q \rightarrow 1$), we have made the assumption that the plots—the Tsallis q -exponential plots—should regain their classical definition. On the other hand, as $q \rightarrow 0$, the curves should deviate away from the classical exponential definition.

Table 1: Miscellaneous quantum-values

q	$\frac{x^{1-q}-1}{1-q}$	$\left\{ \frac{x^{1-q}-1}{1-q} \right\} / t_{(1/2)Oxy22}(s^{-1})$
0.1	0.962	0.428
0.2	0.926	0.412
0.3	0.892	0.396
0.7	0.770	0.342
0.9	0.718	0.319

Figure 1: The q -exponential plots are based on the Tsallis q -logarithmic function. Using several miscellaneous q -values, in particular, 0.1, 0.2, 0.3, 0.7, and 0.9 we plotted $\gamma_0 N_q(t)$ vs. t , where γ_0 is a constant.

The exponential plots in Fig.(1) illustrate a remarkably simple idea that is in alignment with the initial ansatz—the classical form should be recovered in the limiting case. As the q -parameter is varied; for instance, when $q = 0.1$ (red) the exponential plot decayed faster than the other plots. On the contrary, for $q = 0.9$ (black) the decay occurred at a slower rate. This suggests that as $q \rightarrow 0$ radioactive decay occurs faster. In

words, faster decays correspond to more intense radiation[23], which was observed in our plots.

Finally, it is clear from our plots that a variation in the q -parameter (as $q \rightarrow 0$) has the effect of increasing the radioactive intensity. In contrast, as $q \rightarrow 1$ the classical form of the radioactive law is recovered. Therefore, the initial assumptions made with regards to Fig.(1) are justified and validated.

4 Factor-label method

4.1 Unit-factor method

In this Section, we will apply the unit-factor method in [24] to the modified radioactive decay laws in Subsections (3.1) and (3.2), and their classical counterpart. The objective here is to confirm whether or not the units of these equations are invariant under a variation of the q -parameter. The classical form of the radioactive law is given by:

$$N(t) = N_0 \exp(-\lambda t), \quad (4.1)$$

and has units of

$$1 = \cancel{\text{nuclei}} = \cancel{\text{nuclei}} * U. \quad (4.2)$$

The quantum form of the radioactive decay law—based on Tsallis statistics—is given by:

$$\gamma_0 N_q(t) = \tilde{e}_q(-\lambda t) \quad (4.3)$$

and has units of

$$1 = \cancel{\text{nuclei}} = \cancel{\text{nuclei}} * U. \quad (4.4)$$

And lastly, the quantum form of the radioactive decay law—based on Def.(2.5)—is given by:

$$N_{\frac{1}{q}}(t) = N_q(t) = N_0 e_q(-\lambda t) = N_0 e_{\frac{1}{q}}(-\lambda t) \quad (4.5)$$

and has units of

$$1 = \cancel{\text{nuclei}} = \cancel{\text{nuclei}} = \cancel{\text{nuclei}} * U. \quad (4.6)$$

To justify these results, we take a closer look at the exponents. Therefore, for

$$\lambda t = t * \frac{\ln(2)}{t_{(1/2)}}, \quad (4.7)$$

in Eqs.(4.1) and (4.5); the units are

$$1 * (\cancel{\text{seconds}})^{-1} * \cancel{\text{seconds}} = U. \quad (4.8)$$

Here, U is used in commemoration of the dimensionless unit *uno*. Now in terms of the radioactive decay constant, the units for the Tsallis function also remained intact. Thus we have

$$\lambda_q t = \frac{\ln_q(2) * t}{t_{(1/2)Oxy22}} \quad (4.9)$$

with units of,

$$1 * (\cancel{\text{seconds}})^{-1} * \cancel{\text{seconds}} = U. \quad (4.10)$$

Hence, Eqs.(4.8) and (4.10) have identical units. This stems from the fact that the quantum-factorial— $[n]_q!$ —in Def.(2.5) has units of dimension unity.

5 Conclusion

Up until now, we had not discussed a rationale for the use of Tsallis q -logarithm in our analysis. Nevertheless, we were able to use Tsallis q -logarithm along with the property of its inverse function in this work because the solution of the radioactive decay equation has a q -exponential mathematical structure—a growth equation is also applicable here as well. Therefore, we plotted and visualized the behavior of a set of miscellaneous q -values, in the superadditive—superextensive—regime.

In Subsection (3.2) we briefly mentioned Tsallis nonextensive statistical mechanics which was developed on the general principles of Boltzmann-Gibbs (BG) statistical mechanics. The term nonextensive as applied here to nonextensive statistical mechanics arises from the fact that independent systems—for example Q^α and Q^γ —are not additive when $q \neq 1$ [25]. In other words, the probability of two independent systems Q^α and Q^γ , for the entropic parameter $q \neq 1$ are nonadditive. Hence, the name nonextensive—nonadditive—statistical mechanics is given to the theory. The entropic parameter q is a measure of the nonadditivity in the system[26].

Within the framework of Nonadditive statistical mechanics the Boltzmann-Gibbs entropy S_{GB} is extremized and in the limit as the entropic parameter $q \rightarrow 1$, the ordinary Boltzmann-Gibbs entropy is recovered. The entropic parameter q , is determined by the microscopic dynamics of the system and is characterized by the degree of non-additivity. In addition, if Q^α and Q^γ are two independent systems then from probability theory it must be true that $p(Q^\alpha + Q^\gamma) = p(Q^\alpha)p(Q^\gamma)$. But entropy is nonnegative so $S_q \geq 0$ —Tsallis entropy—and hence, for $q < 1$ we have the superadditivity condition or superextensivity[26]. For $q = 1$, we have additivity or extensivity, and finally for $q > 1$ we have subadditivity or subextensivity.

In spite of what was previously mentioned about S_q , we shall elucidate here a bit more on the Tsallis entropy to reinforce the idea of nonadditivity; that is, when $q \neq 1$. Moreover, we will give a short but concrete mathematical description of the intimate connection between S_q and *Jackson's derivative*; that does not appear obvious in the initial stages of this work. Therefore, Tsallis entropy for Q^α and Q^γ is given by

$$S_q(Q^\alpha) = \frac{1 - \sum_i p_{Q^\alpha}^q}{q - 1}, \text{ and} \quad (5.1)$$

$$S_q(Q^\gamma) = \frac{1 - \sum_j p_{Q^\gamma}^q}{q - 1}. \quad (5.2)$$

respectively [27]. This means that,

$$\begin{aligned} S_q(Q^\alpha + Q^\gamma) &= \frac{1 - \sum_i \sum_j p_{Q^\alpha}^q p_{Q^\gamma}^q}{q - 1} \\ &= \frac{2 - \sum_i p_{Q^\alpha}^q - \sum_j p_{Q^\gamma}^q}{q - 1} \\ &= \frac{(1 - \sum_i p_{Q^\alpha}^q)(1 - \sum_j p_{Q^\gamma}^q)}{q - 1} \\ &= \frac{1 - \sum_i p_{Q^\alpha}^q}{q - 1} + \frac{1 - \sum_j p_{Q^\gamma}^q}{q - 1} \\ &\quad - (q - 1) \frac{(1 - \sum_i p_{Q^\alpha}^q)}{q - 1} \frac{(1 - \sum_j p_{Q^\gamma}^q)}{q - 1} \\ &= S_q(Q^\alpha) + S_q(Q^\gamma) - (q - 1)S_q(Q^\alpha)S_q(Q^\gamma). \end{aligned} \quad (5.3)$$

The inspiration behind the development of Tsallis entropy (S_q) is based upon the theory

of multifractal geometry—non-integer representations of fractal dimensions. Nevertheless, Tsallis was able to put forward a concrete generalization of Claude Shannon’s entropy; and later it was S. Abe in [28] who noticed a connection between *Jackson’s derivative* and Tsallis entropy (S_q) in nonextensive statistical mechanics.

To show that there is a connection between nonextensive statistical mechanics and the *Jackson derivative* we begin with the BG theory. Therefore, the BG entropy in discrete form is given by,

$$S_{BG} = - \sum_i p_i \ln p_i. \quad (5.4)$$

Alternatively, the BG entropy in [29] can also be expressed as a continuous function which is given by

$$S_{BG} = - \int dx p(x) \ln p(x). \quad (5.5)$$

However, this link or connection can be deduced by considering Tsallis generalization of the Shannon entropy in [30] which is given by

$$S_q = \frac{1}{q-1} \left(1 - \sum_i p_i^q \right). \quad (5.6)$$

On the other hand, the continuous form of the Tsallis entropy is given by

$$S_q = \frac{1 - \int dx [p(x)]^q}{q-1}. \quad (5.7)$$

Hence, if we apply the definition of the *Jackson derivative* to the second term in parenthesis of Eq.(5.6), we get

$$D_q \sum_i p_i^x = \frac{\sum_i p_i^{qx} - \sum_i p_i^x}{x(q-1)}. \quad (5.8)$$

In the limiting case when $x \rightarrow 1$ we have

$$\lim_{x \rightarrow 1} D_q \sum_i p_i^x = \lim_{x \rightarrow 1} \frac{\sum_i p_i^{qx} - \sum_i p_i^x}{x(q-1)} = -S_q; \quad (5.9)$$

unmasking the connection that Abe noticed between S_q and D_q .

5.1 Concluding remarks

We have shown in our analysis, that by use of the power series method and q -Calculus, it is possible to write a q -analog for the radioactive decay law. Additionally, we extended our analysis and showed that the radioactive decay constant can be written in terms of a q -analog—based on Tsallis statistics—while the units remained invariant in the limit as $q \rightarrow 1$. Furthermore, using Tsallis q -logarithmic function and the radioactive decay constant, we generated some *superextensive plots*. A methodical analysis of the plots—Fig.(1)—found that as $q \rightarrow 0$, the decay process occurred at a faster rate. In other words, the radioactive decay intensity increased.

On the contrary, as $q \rightarrow 1$ the plots regained the classical form of the radioactive decay law. Now for the modified *half-life*, $t_{(1/2)Oxy22} = 2.25$ seconds and $\lambda_q = [(x^{1-q} - 1)/(1 - q)]/t_{(1/2)Oxy22}$, a variation in the q -parameter was akin to the production of a new isotope. This is synonymous with a phenomenon in nuclear physics known as isotopic transmutation—each isotope has its own characteristic radioactive decay constant. Note that while the *half-life* of the isotope—oxygen-22—remained constant, $\ln_q(2)$ changed $\forall q \in (0, 1)$.

In conclusion, we will extend our analysis in Sections (3) and (4) to systems with more than one radioactive decay chain. In particular, we will consider Bateman system of differential equations for stable daughter nuclide—two nuclide in a chain—with $\lambda^d = 0$, $N^d(0) = 0$, and $N^d(t) = N^p(0)(1 - \exp(-\lambda^p t))$. Here, $N^p(0)$ and $N^d(t)$ are the parent and daughter nuclide, respectively.

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