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Title of the paper.

Fundamental Equation of Triangles.

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Note: Some notations in equations may not display correctly in some devices.

Abstract: This paper establishes a **fundamental equation** $a^n + b^n = c^n$, applicable for any triangle, having a, b and c as the sides with 'c' being the longest side and n is a number varying from 1 to infinity. Here, a, b, c and n need not always be integers. It also arrives at a relation between largest angle C (opposite to the longest side 'c') and sides of the triangle with the equation given by,

$$= \cos C = \frac{(1+r^2) - \{(1+r^n)\}^{2/n}}{2r}, \text{ where } r = b/a. \text{ The paper graphically and mathematically}$$

illustrates the relation between the angle C and 'n', for different values of 'n' and 'r' both varying from 1 to infinity. The paper also shows that Pythagoras theorem is a particular case of the above fundamental equation, when n=2. The paper shows that the above fundamental equation is valid even when a, b and c are integers for non-integer values of n>2. This gives a new way of looking at Fermat's Last Theorem.

Keywords: (Geometry, Triangle, Equation, largest angle, ratio of sides.)

1. Introduction

Let us consider Pythagoras theorem. It is about relation between sides of a right angled triangle

Figure-1 shows a right angled triangle in which a=3 units, b=4 units and c=5 units. According to Pythagoras theorem, we have

$$a^2 + b^2 = c^2 \quad (1)$$

Substituting the values,

$$3^2 + 4^2 = 5^2$$

Figure 1. Right angled triangle, illustrating Pythagoras theorem.

In Pythagoras theorem, the numbers are raised to the power $n=2$.

2. Deriving fundamental equation

2.1. Applying cosine rule for right angled triangle.

The basic relation in any triangle with sides a, b, c and angles opposite to the sides are A, B and C respectively (Figure-2) are given by the cosine rule

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad (2)$$

Figure 2. Triangle showing sides and angles for cosine rule.

Let 'a' be the smallest side, and 'c' be the longest side in all the cases considered and hence the largest angle will be angle C.

Putting $a^2 + b^2 = c^2$, in equation (2), we get

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{a^2 + b^2 - a^2 - b^2}{2ab} = 0.$$

So, in any triangle satisfying equation (1), the largest angle will be $\cos^{-1} 0 = 90^\circ$ and it is the case of Pythagoras theorem.

2.2. Putting $n=1$

By substituting $n=1$ in equation (1), it becomes

$$a + b = c,$$

$$\text{Or } c^2 = (a + b)^2 \quad (3)$$

Substituting this in the equation (2), we get

$$\cos C = \frac{a^2 + b^2 - (a + b)^2}{2ab}$$

Expanding the terms,

$$\cos C = \frac{a^2 + b^2 - a^2 - b^2 - 2ab}{2ab} = -1$$

Hence the largest angle will be $= \cos^{-1}(-1) = 180^\circ$

Example: let $a=3$, $b=4$ units. Then, it is represented by a triangle PQR, which is a straight line of length $3+4=7$ units, with angle at $Q = 180^\circ$ and other two angles $=0^\circ$ each, shown in figure 3.

Figure 3. Figure showing triangle becoming a straight line PQR.

2.3. Putting $n=3$.

When $n=3$, equation (1) becomes

$$a^3 + b^3 = c^3 \quad (4)$$

Note: Fermat's last theorem says all the three a, b, c cannot be integers for integers of $n > 2$. But in our case, they not necessarily be integers, as they are representing the sides of a triangle. Here, n also need not be an integer always.

As ' a ' is assumed as smallest of the three numbers, Let the ratio of b/a as ' r '.

Substituting this in equation (4), we get $a^3 + (ar)^3 = c^3$ or,

$$a^3(1 + r^3) = c^3.$$

$$\text{Therefore, } c = \{a^3(1 + r^3)\}^{1/3}$$

On simplifying, $c = a\{(1 + r^3)\}^{1/3}$

$$\text{So, } c^2 = a^2(1 + r^3)^{2/3}$$

Substituting this in equation (2),

$$= \cos C = \frac{a^2(1 + r^2) - a^2\{(1 + r^3)\}^{2/3}}{2a^2r}$$

Dividing numerator and denominator by a^2 , the equation reduces to

$$= \cos C = \frac{(1 + r^2) - \{(1 + r^3)\}^{2/3}}{2r}$$

$$\text{For } r=1, \text{ angle } C = 78.09^\circ$$

So, at **78.09 degrees**, we get $a^3 + b^3 = c^3$

Example: The figure 4 shows a triangle $a=b=2$ as sides for which the closing side is 2.52.

Substituting the values, $2^3 + 2^3 = 8 + 8 = 16 = (2.52)^3$

Figure 4. Triangle showing sides and angle for $n=3$, $r=1$.

2.4. Generalizing for the equation for ' n '.

Generalizing the equation for n , we get

$$= \cos C = \frac{(1 + r^2) - \{(1 + r^n)\}^{2/n}}{2r} \quad (5)$$

This is the general equation, giving the largest angle for n varying from 1 to infinity.

3. Results

In the equation (5), there are **two** variables. They are ' r ' and ' n '. As the side ' a ' is smallest and $r=b/a$, the minimum value of ' r ' is 1 because for $r < 1$, b will become smaller than ' a '. The maximum value of ' r ' is infinity. Similarly, the minimum value of $n=1$ and the maximum value is infinity.

Table1. Relation between characteristic angle and n . ($r=1$).

Serial number	Angle C (in degrees.)	n
1	180	1
2	90	2
3	78.09442	3
4	72.96875	4
5	70.10813	5
6	68.28195	6
7	67.01481	7
8	66.08403	8
9	65.37134	9
10	64.80813	10
11	64.35183	11
12	63.97464	12
13	63.65761	13
14	63.38743	14
15	63.15442	15
16	62.9514	16
17	62.77293	17
18	62.61482	18
19	62.47377	19
20	62.34715	20

Figure 5. Pictorial view of largest angles for some values of n .

Figure 6. Relation between characteristic angle and n for $r=1$

Above graph is drawn from $n=3$, as $n=1$ and 2 angles are same for all values of r

Table2 . Variation of largest angle C for $r=2,3$ and 4 , for $n=1$ to 20 .

n	Angle C for $r=2$	Angle C for $r=3$	Angle C for $r=4$
1	180	180.000000	180.000000
2	90	90.000000	90.000000
3	80.31026	82.53881158	84.0177393
4	77.33657	80.9419074	83.04455015
5	76.25417	80.54920283	82.86434506
6	75.82889	80.44576627	82.82864266
7	75.65417	80.41731695	82.82125838
8	75.58018	80.40925291	82.8196849

9	75.54815	80.40691592	82.81934222
10	75.53404	80.40622708	82.81926634
11	75.52774	80.40602133	82.81924932
12	75.5249	80.4059592	82.81924546
13	75.5236	80.40594028	82.81924458
14	75.523	80.40593447	82.81924437
15	75.52273	80.40593267	82.81924433
16	75.5226	80.40593212	82.81924432
17	75.52254	80.40593194	82.81924431
18	75.52251	80.40593189	82.81924431
19	75.5225	80.40593187	82.81924431
20	75.52249	80.40593187	82.81924431

Angle C varies slightly differently for $r=2, 3$, and 4 for n varying from 3 to 20 .

Figure -7. Variation of angle for various values of n for $r=2, 3$ and 4 .

We have been considering $n=1, 2, 3, \dots$, but ' n ' need not be an integer.

Let us consider some values of ' n ' between 1 and 2 .

Table3 . Largest angles for different values of ' n ' between 1 and 2 .

Serial number	n	Angle C(degrees)
1	1	180.000000
2	1.05	150.719466
3	1.1	139.74557
4	1.15	132.002996
5	1.2	125.972795
6	1.25	121.045363
7	1.3	116.899239
8	1.35	113.339379
9	1.4	110.236757
10	1.45	107.500848
11	1.5	105.065378
12	1.55	102.880226

13	1.6	100.906516
14	1.65	99.1134659
15	1.7	97.4762832
16	1.75	95.9747122
17	1.8	94.5919996
18	1.85	93.3141405
19	1.9	92.1293175
20	1.95	91.0274765
21	2	90.0000001

Figure 8. Pictorial view of largest angle for some values of n between 1 and 2.

Figure 9. Variation of largest angle for values of n between 1 and 2.

Figure 10. Variation of largest angle for values of r between 1 and 10, for $n=5$.

4. Discussion

- 4.1. It is evident that any triangle with sides a , b , c can be expressed as $a^n + b^n = c^n$ for which the characteristic angle C is given by the equation.

$$= \cos C = \frac{(1+r^2) - \{(1+r^n)\}^{2/n}}{2r}, \text{ where } r=b/a.$$

Pythagoras theorem is a particular case of the above equation when $n=2$.

- 4.2. Conversely, we can find a number ' n ' for any given triangle with side a , b and c satisfying the equation $a^n + b^n = c^n$.

Example. Finding ' n ' for a given triangle.

Consider the above triangle of $a=b=100$ and $c=120$. Since $a=b$, $r=1$.

From equation (2), we have

$$= \cos C = \frac{(1+r^2) - \{(1+r^n)\}^{2/n}}{2r},$$

When $r=1$, the equation reduces to

$$\begin{aligned}
&= \cos C = \frac{2-2^{2/n}}{2}, \\
&= 2\cos C - 2 = -2\left(\frac{2}{n}\right),
\end{aligned} \tag{6}$$

From cosine rule, $\cos C = \frac{a^2+b^2-c^2}{2ab}$

Substituting for a, b, c, we get $\cos C = \frac{100^2+100^2-120^2}{2*100*100}$

$C = \cos^{-1} 0.28$. So, angle $= 73.74^\circ$

(The angle can also be directly measured and substituted in the equation, if it is possible to measure it with sufficient accuracy)

Substituting $\cos C = 0.28$ in equation (6), we get

$$= 2 * 0.28 - 2 = -2\left(\frac{2}{n}\right)$$

So, $-1.44 = -2\left(\frac{2}{n}\right)$

Or $(1.44)^{n/2} = 2$

$$n = 2\left(\frac{\log 2}{\log 1.44}\right)$$

Solving, we get $n = 3.802$ (approx)

This is an example where sides a, b, and c are integers where as 'n' is not an integer but it is > 2 .

4.3. As $r \rightarrow \infty$, angle C tends to 90°

From equation(2), we have

$$= \cos C = \frac{(1+r^2)-\{(1+r^n)\}^{2/n}}{2r},$$

In the equation above, for large value of r,

$$1 + r^2 \rightarrow r^2 \text{ and } \{(1 + r^n)\}^{2/n} \rightarrow r^2.$$

Hence the equation simplifies to

$$= \cos C = \frac{r^2 - r^2}{2r} = 0. \quad \text{So, angle } C \rightarrow 90^\circ$$

This is indicated in the figure 10.

4.4. As $n \rightarrow \infty$, largest angle $\rightarrow 60^\circ$ for $r=1$.

When $r=1$, equation (2),

$$= \cos C = \frac{2 - 2^{2/n}}{2},$$

$$\text{For large values of } n, \cos C = \frac{2 - 2^{2/n}}{2} = \frac{2 - 1}{2} = 1/2$$

$$\cos^{-1} 1/2 = 60^\circ$$

This is indicated in the figure 6.

- 4.5.** Only when $n=1$ and $n=2$, all terms in cosine rule cancels out to give angle C as 180° and 90° respectively. Hence the largest angle is independent of side ratios. But for all other values of n , angle C depends upon both r and n .

5. Conclusion

- 5.1.** Fundamental equation of any triangle with sides a, b, c is $a^n + b^n = c^n$, where n is any number between 1 to infinity.
- 5.2.** Pythagoras theorem is a particular case of the above equation for $n=2$.
- 5.3.** The largest angle C opposite to the longest side c can be found out by the general equation $\cos C = \frac{(1+r^2) - \{(1+r^n)\}^{2/n}}{2r}$, where $r=b/a$.
- 5.4.** When ' n ' is not an integer, a, b and c can be integers even for $n>2$. This has been illustrated in 4.4.2 of this paper, providing a new perspective of looking at Fermat's last theorem.
- 5.5.** The fundamental equation of the triangle is very likely to become a stepping stone in finding a simpler proof for Fermat's last theorem.

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Conflict of interest

Author declares no conflict of interest.

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