On the dimension of $H^*((\mathbb{Z}_2)^{\times t}, \mathbb{Z}_2)$ as a module over Steenrod ring

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Abstract

We write $\mathbb{P}$ for the polynomial algebra in one variable over the finite field $\mathbb{Z}_2$ and $\mathbb{P}^{\otimes t} = \mathbb{Z}_2[x_1, \dots, x_t]$ for its t -fold tensor product with itself. We grade $\mathbb{P}^{\otimes t}$ by assigning degree 1 to each generator. We are interested in determining a minimal set of generators for the ring of invariants $(\mathbb{P}^{\otimes t})^{G_t}$ as a module over Steenrod ring, $\mathcal{A}_2$. Here G_t is a subgroup of the general linear group $GL(t, \mathbb{Z}_2)$. An equivalent problem is to find a monomial basis of the space of "unhit" elements, $\mathbb{Z}_2 \otimes_{\mathcal{A}_2} (\mathbb{P}^{\otimes t})^{G_t}$ in each t and degree $n \geq 0$. The structure of this tensor product is proved surprisingly difficult and has been not yet known for $t \geq 5$, even for the trivial subgroup $G_t = \{e\}$. In the present paper, we consider the subgroup $G_t = \{e\}$ for $t \in \{5, 6\}$, and obtain some new results on $\mathcal{A}_2$ -generators of $(\mathbb{P}^{\otimes t})^{G_t}$ in some degrees. At the same time, some of their applications have been proposed. We also provide an algorithm in MAGMA for verifying the results. This study can be understood as a continuation of our recent works in [24, 26].

Keywords:

Steenrod algebra, Peterson hit problem, Adams spectral sequences, Primary cohomology operations, Actions of groups on commutative rings, Algebraic transfer

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(In memory of Professor Reg Wood)

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1. Introduction

This serves as the motivation and introduction of the article. Throughout the context, we fix the prime field of two elements, $\mathbb{Z}_2$. The study of the mod-2 Steenrod algebra $\mathcal{A}_2$ began with Steenrod's work [32] constructing stable cohomology operations acting on cohomology theory with $\mathbb{Z}_2$ -coefficients. This algebra consists of the Steenrod squaring operations Sq^i for $i \geq 0$, which act on the $\mathbb{Z}_2$ -cohomology groups of spaces. These squares are the cohomology operations satisfying the naturality property. Moreover, they commute with the suspension maps and so they are stable. Afterwards, the structure of $\mathcal{A}_2$ was elucidated by Adem [2], Cartan [6] and Serre [29]. It plays a crucial role in the solution of many problems, such as calculating homotopy groups of n -sphere, Hopf invariant one problem, and characteristic classes of vector bundles in Algebraic Topology. We know that $\mathbb{P}^{\otimes t} = \mathbb{Z}_2[x_1, \dots, x_t] = H^*((\mathbb{Z}_2)^{\times t}, \mathbb{Z}_2)$ with $|x_j| = 1$ for every j , and this representation of $\mathbb{P}^{\otimes t}$ defines it as an $\mathcal{A}_2$ -module. The polynomial ring $\mathbb{P}^{\otimes t}$ is a connected $\mathbb{Z}$ -graded algebra. One of central problems of Algebraic Topology is to determine a minimal set of generators for the ring of invariants $(\mathbb{P}^{\otimes t})^{G_t}$, where $G_t \subseteq GL_t := GL(t, \mathbb{Z}_2)$, the general linear group of invertible matrices. This $(\mathbb{P}^{\otimes t})^{G_t}$ is a graded $\mathcal{A}_2$ -submodule of the left $\mathcal{A}_2$ -module $\mathbb{P}^{\otimes t}$. Equivalently, we need to find the dimension of the following $\mathbb{Z}$ -graded vector space over $\mathbb{Z}_2$:

$$(1.1) \quad \{(\mathbb{Z}_2 \otimes_{\mathcal{A}_2} (\mathbb{P}^{\otimes t})^{G_t})_n\}_{n \in \mathbb{Z}, n \geq 0}.$$

This problem is generally regarded as difficult for arbitrary number t of variables and it has been of interest to topologist (see Hung-Peterson [11], Kameko [13], Mothebe-Uys [17], Nam [18], Pengelley-Williams [19], Peterson [20, 21], the present writer [22, 23, 24, 25, 26, 27], Singer [30, 31], Sum [33, 34, 35, 36], Walker-Wood [37] and others). Alternatively, its applications to homotopy theory are surveyed by Peterson [21] and Singer [30]. For $G_t = \Sigma_t$ (the symmetric group on t letters), the space (1.1) is known by Singer's work [31] for the dual of (1.1) with $t \geq 0$. In the case in which $G_t = GL_t$, the problem is studied by Singer [30] for $t = 2$, and by Hung-Peterson [11] for $t = 3, 4$. Up to present, it remains undetermined in general. In the case in which $G_t = \{e\}$, the trivial subgroup, one has

$$(1.2) \quad \{Q_n^{\otimes t} := (\mathbb{Z}_2 \otimes_{\mathcal{A}_2} \mathbb{P}^{\otimes t})_n\}_{n \in \mathbb{Z}, n \geq 0},$$

as a modular representation of GL_t . Studying this space is related to the E_2 -page of the Adams spectral sequence for the stable homotopy groups of spheres, which is mentioned below. The structure of the space (1.2) was explicitly determined by Peterson [20] for $t = 1, 2$, by Kameko [13] for $t = 3$, by Sum [34] for $t = 4$. The case $t = 5$ has been investigated by the author [22, 23, 24, 25, 26, 27], and Sum [35] in some certain degrees. But when $t \geq 6$, the answer is not yet known. The primary interest of this work is the cases $t = 5$ and 6, as well as their applications. There exist a few approaches to reduce the computation of the dimension of the space (1.2). The squaring operation $\widetilde{Sq}_*^0 : Q_{t+2n}^{\otimes t} \longrightarrow Q_n^{\otimes t}$, which was constructed by Kameko [13], could be one of those useful approaches. One should note that this $\widetilde{Sq}_*^0$ is an epimorphism of $\mathbb{Z}_2 GL_t$ -modules. The μ -function is also one of the arithmetic functions that have much been used in this context (see also [13], [35]). The range of it is the minimum number of terms of the form $2^d - 1$, $d > 0$, with repetitions allowed, whose sum is n , from which due to Wood [38], $Q_n^{\otimes t}$ is trivial if $\mu(n) > t$, and due to Kameko [13], the Kameko $\widetilde{Sq}_*^0$ is an isomorphism if $\mu(t + 2n) = t$. In general, it is not easy to compute or even estimate the dimension of $Q_n^{\otimes t}$ in all degrees n . However, as the results of Wood [38] and Kameko [13] mentioned, the dimension of the space (1.2) can be completely determined in each n of the so-called "generic degree":

$$(1.3) \quad n = r(2^d - 1) + m \cdot 2^d,$$

whenever $0 \leq \mu(m) < r < t$, and $d \geq 0$. As above shown, the problem we are interested in is systematically studied when $t \leq 4$, however it remains a mystery in the general case. Moreover, there is no general rule for studying the problem in each t and degree n of the form (1.3) above, that means each computation is important on its own. Therefore, our motivation to write up this article is to probe the space (1.2) and its applications for the cases $t = 5$ and 6 . Before presenting the main results, quoted below in detail, we need some background material for completeness of the exposition and for the reader's convenience. In that aspect, the next four chief concepts can be found in literatures [13], [24], [34], [37].

Definition 1.1 (Weight vector and exponent vector). A weight vector ω is a sequence of non-negative integers $\omega = (\omega_1, \omega_2, \dots, \omega_j, \dots)$ such that $\omega_j = 0$ for $j \gg 0$. Let $X = x_1^{u_1} x_2^{u_2} \dots x_t^{u_t} \in \mathbb{P}_n^{\otimes t}$, then one defines the *weight vector* and the *exponent vector* of X by

$$\omega(X) := (\omega_1(X) = \sum_{1 \leq j \leq t} \alpha_0(u_j), \omega_2(X) = \sum_{1 \leq j \leq t} \alpha_1(u_j), \dots, \omega_m(X) = \sum_{1 \leq j \leq t} \alpha_{m-1}(u_j), \dots)$$

and $u(X) := (u_1, u_2, \dots, u_t)$, respectively, where $\alpha_j(n) \in \{0, 1\}$ denotes the j -th coefficients in dyadic expansion of a natural number n . One also defines $\deg(\omega(X)) = \sum_{j \geq 1} 2^{j-1} \omega_j(X)$ and by convention, the sets of all weight vectors and exponent vectors are given the left lexicographical order.

From now on, let us write $\mathbb{P}_n^{\otimes t}$ for the subspace of $\mathbb{P}^{\otimes t}$ generated by the homogeneous polynomials of degree n in $\mathbb{P}^{\otimes t}$. Then $\mathbb{P}^{\otimes t} = \{\mathbb{P}_n^{\otimes t}\}_{n \in \mathbb{Z}, n \geq 0}$, in which $\mathbb{P}_0^{\otimes t} \cong \mathbb{Z}_2$. It is known that by restricting to non-singular matrices, $\mathbb{P}_n^{\otimes t}$ also gives a modular representation of GL_t , and the Steenrod squaring operations $Sq^i : \mathbb{P}_{n-i}^{\otimes t} \rightarrow \mathbb{P}_n^{\otimes t}$ are maps of $\mathbb{Z}_2 GL_t$ -modules.

Definition 1.2 (Linear order on $\mathbb{P}^{\otimes t}$). Assume that $X = x_1^{u_1} x_2^{u_2} \dots x_t^{u_t}$ and $Y = x_1^{v_1} x_2^{v_2} \dots x_t^{v_t}$ are the monomials in $\mathbb{P}_n^{\otimes t}$. We say that $X < Y$ if and only if one of the following holds:

- (i) $\omega(X) < \omega(Y)$;
- (ii) $\omega(X) = \omega(Y)$ and $u(X) < v(Y)$.

Definition 1.3 (Equivalence relations on $\mathbb{P}^{\otimes t}$). For a weight vector ω of degree n , we denote two subspaces associated with ω by

$$\begin{aligned} (\mathbb{P}_n^{\otimes t})^{\leq \omega} &= \langle \{X \in \mathbb{P}_n^{\otimes t} \mid \deg(X) = \deg(\omega) = n, \omega(X) \leq \omega\} \rangle, \\ (\mathbb{P}_n^{\otimes t})^{< \omega} &= \langle \{X \in \mathbb{P}_n^{\otimes t} \mid \deg(X) = \deg(\omega) = n, \omega(X) < \omega\} \rangle. \end{aligned}$$

Given homogeneous polynomials F and G in $\mathbb{P}_n^{\otimes t}$, one defines the equivalence relations " $\equiv$ " and " $\equiv_\omega$ " on $\mathbb{P}^{\otimes t}$ by stating that

- (i) $F \equiv G$ if and only if $(F - G) \in \overline{\mathcal{A}_2} \mathbb{P}_n^{\otimes t}$;
- (ii) $F \equiv_\omega G$ if and only if $F, G \in (\mathbb{P}_n^{\otimes t})^{\leq \omega}$ and $(F - G) \in ((\overline{\mathcal{A}_2} \mathbb{P}_n^{\otimes t} \cap (\mathbb{P}_n^{\otimes t})^{\leq \omega} + (\mathbb{P}_n^{\otimes t})^{< \omega}), \overline{\mathcal{A}_2} \mathbb{P}_n^{\otimes t} = \sum_{i \geq 0} \text{Im}(Sq^{2^i})$ denoting the set of all the $\mathcal{A}_2$ -decomposable elements, in which $Sq^{2^i} : \mathbb{P}_{n-2^i}^{\otimes t} \rightarrow \mathbb{P}_n^{\otimes t}$ and $\overline{\mathcal{A}_2}$ is the kernel of an epimorphism $\mathcal{A}_2 \rightarrow \mathbb{Z}_2$ of graded $\mathbb{Z}_2$ -algebras. In particular, if F is a linear combination of elements in the images of the Steenrod squaring operations Sq^{2^i} for $i \geq 0$, then we say that F is *decomposable* (or *hit*). In other words, $F \equiv 0$. In addition, F is said to be ω -*decomposable* if $F \equiv_\omega 0$. Then, we have a $\mathbb{Z}_2$ -quotient space of $Q^{\otimes t}$ by the equivalence relation " $\equiv_\omega$ ": $(Q_n^{\otimes t})^\omega = (\mathbb{P}_n^{\otimes t})^{\leq \omega} / ((\overline{\mathcal{A}_2} \mathbb{P}_n^{\otimes t} \cap (\mathbb{P}_n^{\otimes t})^{\leq \omega}) + (\mathbb{P}_n^{\otimes t})^{< \omega})$.

Due to Sum [36], $(Q_n^{\otimes t})^\omega$ is also an $\mathbb{Z}_2 GL_t$ -module for every n . The above definitions lead to the following that will be played a central role in the content of this text.

Definition 1.4 (Admissible monomial and inadmissible monomial). We say that a monomial $X \in \mathbb{P}_n^{\otimes t}$ is *inadmissible* if there exist monomials $Y_1, Y_2, \dots, Y_k$ such that $Y_j < X$ for $1 \leq j \leq k$ and $X \equiv \sum_{1 \leq j \leq k} Y_j$. Then, X is said to be *admissible* if it is not inadmissible.

It is important to remark that the set of all the admissible monomials in $\mathbb{P}_n^{\otimes t}$ is a *minimal set of $\mathcal{A}_2$ -generators* for $\mathbb{P}^{\otimes t}$ in degree n . Therefore, $Q_n^{\otimes t}$ is an $\mathbb{Z}_2$ -vector space with a basis consisting of all the classes represent by the elements in $\mathbb{P}_n^{\otimes t}$. Specifically, by our previous discussions [24], $\dim Q_n^{\otimes t}$ can be represented as $\sum_{\deg(\omega)=n} \dim(Q_n^{\otimes t})^\omega$. We fix some notations to be used through the paper. Writing $(\mathbb{P}^{\otimes t})^0$ and $(\mathbb{P}^{\otimes t})^{>0}$ for $\mathcal{A}_2$ -submodules of $\mathbb{P}^{\otimes t}$ spanned all the monomials $x_1^{a_1} x_2^{a_2} \dots x_t^{a_t}$ such that $\prod_{1 \leq j \leq t} a_j = 0$, and $\prod_{1 \leq j \leq t} a_j > 0$, respectively. Writing $(\mathbb{P}_n^{\otimes t})^0 := ((\mathbb{P}^{\otimes t})^0)_n$ and $(\mathbb{P}_n^{\otimes t})^{>0} := ((\mathbb{P}^{\otimes t})^{>0})_n$ for the homogeneous components of degrees n in $(\mathbb{P}^{\otimes t})^0$ and $(\mathbb{P}^{\otimes t})^{>0}$, respectively. We put $(Q_n^{\otimes t})^0 := (\mathbb{Z}_2 \otimes_{\mathcal{A}_2} (\mathbb{P}^{\otimes t})^0)_n$, and $(Q_n^{\otimes t})^{>0} := (\mathbb{Z}_2 \otimes_{\mathcal{A}_2} (\mathbb{P}^{\otimes t})^{>0})_n$, from which one has that $Q_n^{\otimes t} = (Q_n^{\otimes t})^0 \oplus (Q_n^{\otimes t})^{>0}$.

We are now able to give the main results of this work. Our first goal is to investigate the space (1.2) for the case $t = 5$ and the degree of the form (1.3) when $r = t = 5, m = 13$ and $d = 1$. The case $d > 1$ will be discussed at the end of this section. For a start we have seen that due to Kameko's homomorphism, the map $\widetilde{Sq}_*^0 : Q_{5(2^1-1)+13, 2^1}^{\otimes 5} \rightarrow Q_{5(2^0-1)+13, 2^0}^{\otimes 5}$ is an epimorphism of $\mathbb{Z}_2 GL_5$ -modules. On the other side, in our previous work, we have indicated that

Theorem 1.5 (see [24]). *Let $n = 3(2^d - 1) + 2^d$ be the generic degree of form (1.3) with d a positive integer. The dimension of the $\mathbb{Z}_2$ -vector space $Q_n^{\otimes 5}$ is determined by the following table:*

$n = 3(2^d - 1) + 2^d$	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d \geq 6$
$\dim Q_n^{\otimes 5}$	46	250	645	945	1115	1116

So, by the theorem, we only need to determine the kernel of $\widetilde{Sq}_*^0$. For convenience, let us denote $(Q_n^{\otimes t})^{\omega^0} := (Q_n^{\otimes t})^\omega \cap (Q_n^{\otimes t})^0$, and $(Q_n^{\otimes t})^{\omega^{>0}} := (Q_n^{\otimes t})^\omega \cap (Q_n^{\otimes t})^{>0}$. One of our main results is then as follows.

Theorem 1.6. *Consider the weight vectors of degree $31 = 5(2^1 - 1) + 13 \cdot 2^1$:*

$$\omega_{(1)} := (1, 1, 1, 1, 1), \quad \omega_{(2)} := (3, 2, 2, 2), \quad \omega_{(3)} := (3, 4, 3, 1).$$

Then, we have an isomorphism $\text{Ker}(\widetilde{Sq}_^0) \cong (Q_{31}^{\otimes 5})^0 \oplus \left(\bigoplus_{1 \leq j \leq 3} (Q_{31}^{\otimes 5})^{\omega_{(j)}^{>0}} \right)$.*

One can easily compute $(Q_{31}^{\otimes 5})^0$ by using the known results of $Q_{31}^{\otimes 4}$. Nevertheless, for the convenience of the reader, it is detailed as follows. The reader already knows (see also [26]) that for each non-negative integer n , there is an isomorphism $(Q_n^{\otimes 5})^0 \cong \bigoplus_{1 \leq s \leq 4} \bigoplus_{\ell(\mathcal{J})=s} (Q_n^{\otimes \mathcal{J}})^{>0}$, where $Q^{\otimes \mathcal{J}} = \langle [x_{j_1}^{t_1} x_{j_2}^{t_2} \dots x_{j_s}^{t_s}] \mid t_i \in \mathbb{N}, i = 1, 2, \dots, s \rangle \subset Q^{\otimes 5}$ in which $\mathcal{J} = (j_1, j_2, \dots, j_s)$, $1 \leq j_1 < \dots < j_s \leq 5$, $1 \leq s \leq 4$, and $\ell(\mathcal{J}) := s$ denotes the length of $\mathcal{J}$. So, $\dim(Q_n^{\otimes 5})^0 = \sum_{1 \leq s \leq 4} \binom{5}{s} \dim(Q_n^{\otimes s})^{>0}$. Moreover, by the previous works of Peterson [20], Kameko [13] and Sum [34], it follows that

$$\dim(Q_{31}^{\otimes s})^{>0} = \begin{cases} 1, & \text{if } s = 1, \\ 1, & \text{if } s = 2, \\ 8, & \text{if } s = 3, \\ 47, & \text{if } s = 4. \end{cases}$$

We thus have $\dim(Q_{31}^{\otimes 5})^0 = 1 \times \binom{5}{1} + 1 \times \binom{5}{2} + 8 \times \binom{5}{3} + 47 \times \binom{5}{4} = 330$. Next, the dimension results of $(Q_{31}^{\otimes 5})^{\omega_{(j)}^{>0}}$ are determined by the following theorem.

Theorem 1.7. *Let $\omega_{(j)}, j = 1, 2, 3$, be the weight vectors as in Theorem 1.6. Then, with notations as above, we assert that*

$$\dim(Q_{31}^{\otimes 5})^{\omega_{(j)}^{>0}} = \begin{cases} 1, & \text{if } j = 1, \\ 215, & \text{if } j = 2, \\ 70, & \text{if } j = 3. \end{cases}$$

In order to compute the subspaces $(Q_{31}^{\otimes 5})^{\omega_{(j)}^{>0}}$, we adopt some homomorphisms in Sum [34] and a recent result of Mothebe-Uys [17]. This approach allows us to reduce many computations for a monomial basis of the space (1.2) in each certain positive degree. This is shown specifically in this paper by proving Propositions 2.2.4, 2.2.6 in section two and Theorem 1.9 below. As consequences of the results above, we immediately obtain

Corollary 1.8. *There exist exactly 866 admissible monomials of degree 31 in $\mathbb{P}^{\otimes 5}$. Consequently, the $\mathbb{Z}_2$ -vector space $Q_{31}^{\otimes 5}$ is 866-dimensional.*

We continue by considering the case $r = t = 5, m = 32$ and $d = 0$. So, our second goal is to establish the following.

Theorem 1.9. *We have an isomorphism $Q_{32}^{\otimes 5} \cong (Q_{32}^{\otimes 5})^{\omega'_{(1)}} \oplus (Q_{32}^{\otimes 5})^{\omega'_{(2)}} \oplus (Q_{32}^{\otimes 5})^{\omega'_{(3)}}$, where $\omega'_{(1)} := (2, 1, 1, 1, 1)$, $\omega'_{(2)} := (4, 2, 2, 2)$, and $\omega'_{(3)} := (4, 4, 3, 1)$ are the weight vectors of degree 32. Consequently, $Q_{32}^{\otimes 5}$ has dimension 1004.*

From Theorems 1.6 and 1.9, we remark that the dimensions of $Q_{31}^{\otimes 5}$ and $Q_{32}^{\otimes 5}$ are not only large but also complicated in the calculations. So, we have provided an algorithm in MAGMA for verifying these results (see Appendix). Moreover, this algorithm can be used to verify the results of Peterson [20], Kameko [13] and Sum [34] on the dimension of the space (1.2) for the cases $t \leq 4$. Before presenting the next main result, we briefly review Sum's conjecture [34], which provides a connection between admissible monomials in $\mathbb{P}_n^{\otimes(t-1)}$ and $\mathbb{P}_n^{\otimes t}$. For each $1 \leq l \leq t$, define the $\mathbb{Z}_2$ -homomorphism $\mathbf{q}_{(l,t)} : \mathbb{P}^{\otimes(t-1)} \rightarrow \mathbb{P}^{\otimes t}$ by setting that $\mathbf{q}_{(l,t)}(x_j)$ is x_j if $1 \leq j \leq l-1$, and is x_{j+1} if $l \leq j \leq t-1$. We put $\mathcal{N}_t := \{(l, \mathcal{L}) \mid \mathcal{L} = (l_1, l_2, \dots, l_r), 1 \leq l < l_1 < l_2 < \dots < l_r \leq t, 0 \leq r \leq t-1\}$, where by convention, $\mathcal{L} = \emptyset$ if $r = 0$. Writing $r = \ell(\mathcal{L})$ for the length of $\mathcal{L}$. Let us consider the pair $(l, \mathcal{L}) \in \mathcal{N}_t$, $1 \leq r \leq t-1$ and for each $1 \leq u < r$, we set $X_{(\mathcal{L}, u)} = x_{l_u}^{2^{r-1}+2^{r-2}+\dots+2^{r-u}} \prod_{u < d \leq r} x_{l_d}^{2^{r-d}}$ and $X_{(\emptyset, 1)} = 1$. Then, one has the following important linear transformation, which is due to Sum [34]:

$$\begin{aligned} \psi_{(l, \mathcal{L})} : \mathbb{P}^{\otimes(t-1)} &\longrightarrow \mathbb{P}^{\otimes t} \\ \prod_{1 \leq s \leq t-1} x_s^{a_s} &\longmapsto \begin{cases} (x_l^{2^{r-1}} \mathbf{q}_{(l,t)}(\prod_{1 \leq s \leq t-1} x_s^{a_s})) / X_{(\mathcal{L}, u)} & \text{if there exists } u \text{ such that:} \\ & a_{l_1-1} + 1 = \dots = a_{l_{(u-1)}-1} + 1 = 2^r, \\ & a_{l_u-1} + 1 > 2^r, \\ & \alpha_{r-d}(a_{l_u-1}) - 1 = 0, \quad 1 \leq d \leq u, \\ & \alpha_{r-d}(a_{l_d-1}) - 1 = 0, \quad u+1 \leq d \leq r, \\ & \text{otherwise.} \end{cases} \\ &0 \end{aligned}$$

A few comments below are necessary and useful.

Remark 1.10. If $\mathcal{L} = \emptyset$, then $\psi_{(l, \mathcal{L})} = \mathbf{q}_{(l,t)}$ for all $1 \leq l \leq t$. Also, one can easily check that if $\psi_{(l, \mathcal{L})}(\prod_{1 \leq s \leq t-1} x_s^{a_s}) \neq 0$, then $\omega(\psi_{(l, \mathcal{L})}(\prod_{1 \leq s \leq t-1} x_s^{a_s})) = \omega(\prod_{1 \leq s \leq t-1} x_s^{a_s})$. Note that, in general, $\psi_{(l, \mathcal{L})}$ is not a homomorphism of $\mathcal{A}_2$ -modules. An illustrative instance is necessary for this: Taking $t = 4$, $\mathcal{L} = (2, 3, 4) \neq \emptyset$, and the monomial $x_1^{12} x_2^6 x_3^9 \in \mathbb{P}_{27}^{\otimes 3}$, then, it is straightforward to see that

$$\psi_{(1, \mathcal{L})}(x_1^{12} x_2^6 x_3^9) = \frac{x_1^{2^3-1} \mathbf{q}_{(1,4)}(x_1^{12} x_2^6 x_3^9)}{X_{(\mathcal{L}, 1)}} = \frac{x_1^7 x_2^{12} x_3^6 x_4^9}{x_2^4 x_3^2 x_4} = x_1^7 x_2^8 x_3^4 x_4^8 \in \mathbb{P}_{27}^{\otimes 4},$$

and $\omega(x_1^{12} x_2^6 x_3^9) = (1, 1, 2, 2) = \omega(\psi_{(1, \mathcal{L})}(x_1^{12} x_2^6 x_3^9))$. Hence, one gets

$$Sq^2(\psi_{(1, \mathcal{L})}(x_1^{12} x_2^6 x_3^9)) = x_1^9 x_2^8 x_3^4 x_4^8 \neq x_1^7 x_2^8 x_3^6 x_4^8 = \psi_{(1, \mathcal{L})}(Sq^2(x_1^{12} x_2^6 x_3^9)).$$

Now, for a subset $\mathcal{V} \subset \mathbb{P}_n^{\otimes(t-1)}$, let us write

$$\begin{aligned} \widetilde{\Phi}^0(\mathcal{V}) &= \bigcup_{1 \leq l \leq t} \psi_{(l, \emptyset)}(\mathcal{V}) = \bigcup_{1 \leq l \leq t} \mathbf{q}_{(l,t)}(\mathcal{V}), \\ \widetilde{\Phi}^{>0}(\mathcal{V}) &= \bigcup_{(l, \mathcal{L}) \in \mathcal{N}_t, 1 \leq \ell(\mathcal{L}) \leq t-1} (\psi_{(l, \mathcal{L})}(\mathcal{V}) \setminus (\mathbb{P}_n^{\otimes t})^0), \end{aligned}$$

and put $\widetilde{\Phi}_*(\mathcal{V}) = \widetilde{\Phi}^0(\mathcal{V}) \cup \widetilde{\Phi}^{>0}(\mathcal{V})$. It is straightforward to check that $\mathbf{q}_{(l,t)}$ is also an $\mathcal{A}_2$ -homomorphism, which implies that if $\mathcal{V}$ is the set of all the admissible monomials in $\mathbb{P}_n^{\otimes(t-1)}$, then $\widetilde{\Phi}^0(\mathcal{V})$ is the set of all the admissible monomials in $(\mathbb{P}_n^{\otimes t})^0$. Let us now denote by $\mathcal{C}_n^{\otimes t}$ the set of all admissible monomials in $\mathbb{P}_n^{\otimes t}$. If ω is a weight vector of degree n , then we set $(\mathcal{C}_n^{\otimes t})^\omega := \mathcal{C}_n^{\otimes t} \cap (\mathbb{P}_n^{\otimes t})^{\leq \omega}$. In [35], Sum sets up the following yet left-open.

Conjecture 1.11. *Under the above notations, for each $(l, \mathcal{L}) \in \mathcal{N}_t$, and $1 \leq r = \ell(\mathcal{L}) \leq t-1$, if $X = \prod_{1 \leq j \leq t-1} x_j^{a_j} \in \mathcal{C}_n^{\otimes(t-1)}$, and there exist u , $1 \leq u \leq r$ such that*

$$(1.4) \quad \begin{aligned} a_{l_1-1} + 1 = \dots = a_{l_{(u-1)}-1} + 1 = 2^r, \quad a_{l_u-1} + 1 > 2^r, \\ \alpha_{r-d}(a_{l_u-1}) - 1 = 0, \text{ for } 1 \leq d \leq u, \text{ and } \alpha_{r-d}(a_{l_d-1}) - 1 = 0, \text{ for } u+1 \leq d \leq r, \end{aligned}$$

then $\psi_{(l, \mathcal{L})}(X) \in \mathcal{C}_n^{\otimes t}$. Moreover, if ω is a weight vector of degree n , then $\widetilde{\Phi}_((\mathcal{C}_n^{\otimes(t-1)})^\omega) \subseteq (\mathcal{C}_n^{\otimes t})^\omega$.*

Basing the proofs of Theorems 1.7 and 1.9, we have seen that Conjecture 1.11 also holds when $t = 5$ and the degrees 31, 32 (see Remarks 2.2.5 and 2.3.2 in Sect.2).

We now describe the next main result by studying the tensor product $Q^{\otimes 6}$ in generic degree $5(2^{d+4} - 1) + 47 \cdot 2^{d+4}$. To do this, we use our previous results in [26, 27] and get the following.

Theorem 1.12. *Consider degree n_d of the form (1.3) with $r = t = 5$ and $m = 21$. Then, we have that $\dim Q_{5(2^{d+4}-1)+n_1 \cdot 2^{d+4}}^{\otimes 6} = (2^6 - 1) \dim Q_{n_d}^{\otimes 5}$. Consequently, $Q_{5(2^{d+4}-1)+n_1 \cdot 2^{d+4}}^{\otimes 6}$ is 119322-dimensional, for all positive integers d .*

An application to Singer's algebraic transfer. As is well known, computing explicitly mod-2 cohomology groups of the Steenrod algebra, $\text{Ext}_{\mathcal{A}_2}^{t,t+*}(\mathbb{Z}_2, \mathbb{Z}_2)$ is a major open problem of Algebraic Topology, because it is the E_2 -page of the Adams spectral sequence converging to the stable homotopy groups of the spheres. One approach to better understand the structure of these Ext groups was proposed by Singer [30] where he introduced a transfer homomorphism of algebras

$$Tr_{\mathcal{A}_2}^{\mathcal{A}_2} := \bigoplus_{t \geq 0} Tr_t^{\mathcal{A}_2} : \bigoplus_{t \geq 0} [P_{\mathcal{A}_2}((\mathbb{P}^{\otimes t})^*)]_{GL_t} \rightarrow \bigoplus_{t \geq 0} \text{Ext}_{\mathcal{A}_2}^{t,t+*}(\mathbb{Z}_2, \mathbb{Z}_2) = \bigoplus_{t \geq 0} H^{t,t+*}(\mathcal{A}_2, \mathbb{Z}_2)$$

from the subspace of the divided power algebra $(\mathbb{P}^{\otimes t})^* = H_*((\mathbb{Z}/2)^{\times t}, \mathbb{Z}/2)$ consisting of all elements that are $\mathcal{A}_2$ -annihilated by all positive degree Steenrod squaring operations to the cohomology of the Steenrod algebra. It should be noted that the action of the algebra $\mathcal{A}_2$ and the action of the group GL_t on $(\mathbb{P}^{\otimes t})^*$ commute, and so there exists an induced action of GL_t on $P_{\mathcal{A}_2}((\mathbb{P}^{\otimes t})^*)$. As it was demonstrated in the works by Singer [30] for $t \leq 2$ and Boardman [3] for $t = 3$ that $Tr_t^{\mathcal{A}_2}$ is an isomorphism. So, the algebraic transfer is highly nontrivial. Additionally, by the work of Minami [16], Singer's transfer is related to the problem of finding permanent cycles in the classical Adams spectral sequence. It may need to be recalled that the Hopf elements are represented by the elements $h_i \in \text{Ext}_{\mathcal{A}_2}^{1,2^i}(\mathbb{Z}_2, \mathbb{Z}_2)$ on the E_2 -page of the Adams spectral sequence. By Adam's solution of the Hopf invariant one problem [1], only h_i , for $0 \leq i \leq 3$, survive to the E_∞ -page. By Browder's work [4], the Kervaire classes $\theta_i \in \pi_{2^{i+1}-2}^S$, if they exist, are represented by the elements $h_i^2 \in \text{Ext}_{\mathcal{A}_2}^{2,2^{i+1}}(\mathbb{Z}_2, \mathbb{Z}_2)$ on E_2 -page. People have been known that h_i^2 survives for $i \leq 5$, but when $i \geq 7$, the surprising work of Hill, Hopkins, and Ravenel [10] proved that those elements cannot survive to the E_∞ -page. The case $i = 6$ is unknown. Thus, understanding hit problems and Singer's algebraic transfer is quite important for solving the open problems mentioned. Now coming back to the Singer algebraic transfer on hand, the behavior of $Tr_t^{\mathcal{A}_2}$ in higher ranks was partly surveyed by Singer himself [30], where he applies the invariant theory to show that $Tr_4^{\mathcal{A}_2}$ is an isomorphism in the certain internal degrees. Thence, he predicts that $Tr_t^{\mathcal{A}_2}$ is a monomorphism, but this is currently no answer for $t \geq 5$. This hypothesis has been considered by many researchers, including as well the present writer. For instance, due to Singer himself [30] and Boardman [3], the conjecture was confirmed for ranks ≤ 3 . The recent works of the author [23, 24, 26, 27, 28] and Sum [35, 36] have studied it in the ranks 4 and 5 and some certain generic degrees of the form (1.3). Most notably, our calculations [28] have shown that Singer's conjecture holds also for rank four. Continuing these works, in the present paper, we would like to probe this conjecture for the rank 5 case and the internal degrees 14, 31 and 32. To accomplish this, we first explicitly determine the domain of the fifth transfer in those degrees. Therefrom, together with the calculations of Lin [15] and Chen [7] on the fifth cohomology groups of the Steenrod algebra, we obtain the information on $Tr_5^{\mathcal{A}_2}$ in degrees given. More precisely, by direct computations using a monomial basis of $Q_{14}^{\otimes 5}$ (see

Remark 2.5.2 in Sect.2) and the proofs of Theorems 1.7 and 1.9, one gets the technical theorem below.

Theorem 1.13. *With the notations as above,*

$$\dim[P_{\mathcal{A}_2}((\mathbb{P}_n^{\otimes 5})^*)]_{GL_5} = \begin{cases} 1 & \text{if } n = 14, \\ 2 & \text{if } n = 31, \\ 0 & \text{if } n = 32. \end{cases}$$

On the other side, an important result due to Lin [15] and Chen [7] shows that

$$\text{Ext}_{\mathcal{A}_2}^{5,5+n}(\mathbb{Z}_2, \mathbb{Z}_2) = \begin{cases} \langle h_0 d_0 \rangle & \text{if } n = 14, \\ \langle h_0^4 h_5, n_0 \rangle & \text{if } n = 31, \\ \langle h_4 e_0 \rangle = 0 & \text{if } n = 32. \end{cases}$$

Moreover, following Singer [30], Hà [9], Chơn-Hà [8], the non-zero elements $h_i \in \text{Ext}_{\mathcal{A}_2}^{1,2^i}(\mathbb{Z}_2, \mathbb{Z}_2)$ for $i \geq 0$, $d_0 \in \text{Ext}_{\mathcal{A}_2}^{4,4+14}(\mathbb{Z}_2, \mathbb{Z}_2)$ and $n_0 \in \text{Ext}_{\mathcal{A}_2}^{5,5+31}(\mathbb{Z}_2, \mathbb{Z}_2)$, are detected by the algebraic transfer. Also note that although the authors [8] can easily deduce that the fifth transfer is an epimorphism in the internal degree 31, their techniques were not directly applicable in confirming or refuting Singer's hypothesis in this case. So, the following assertion, which is much more important and meaningful than their work, is a direct consequence of the above data, Theorem 1.13 and the fact that $Tr^{\mathcal{A}_2}$ is a homomorphism of algebras.

Corollary 1.14. *The transfer homomorphism*

$$Tr_5^{\mathcal{A}_2} : [P_{\mathcal{A}_2}((\mathbb{P}_n^{\otimes 5})^*)]_{GL_5} \rightarrow \text{Ext}_{\mathcal{A}_2}^{5,5+n}(\mathbb{Z}_2, \mathbb{Z}_2)$$

is an isomorphism for $n \in \{14, 31, 32\}$. Consequently, Singer's conjecture holds for the fifth transfer in those internal degrees.

Discussions and future research. Using the method outlined in the article, we will continue to study the space (1.2) for $t \in \{5, 6\}$, and extend Corollary 1.8 for degree $n_d := 5(2^d - 1) + 13 \cdot 2^d$ whenever $d > 1$. In actual fact, we only need to investigate the case $d = 2$. Indeed, it can be easily seen that $\mu(n_d) = 5$ for any $d > 2$, and we therefore deduce an isomorphism $(\widetilde{Sq}_*^0)^{d-2} : Q_{n_d}^{\otimes 5} \xrightarrow{\cong} Q_{n_2}^{\otimes 5}$, for all $d \geq 2$. This together with Corollary 1.8 allow us to assert the above request. Now, when $d = 2$, notice that the Kameko map $[\widetilde{Sq}_*^0]_{n_2} := \widetilde{Sq}_*^0 : Q_{n_2}^{\otimes 5} \rightarrow Q_{31}^{\otimes 5}$ is an epimorphism, which implies that $Q_{n_2}^{\otimes 5} \cong \text{Ker}[\widetilde{Sq}_*^0]_{n_2} \oplus Q_{31}^{\otimes 5}$. So, invoking Corollary 1.8, we need to compute the kernel of $[\widetilde{Sq}_*^0]_{n_2}$. Using Theorem 1.9 together with some of the same arguments as in the proof of Theorem 1.6, it may be concluded that

$$\text{Ker}[\widetilde{Sq}_*^0]_{n_2} \cong (Q_{n_2}^{\otimes 5})^0 \bigoplus \left(\bigoplus_{1 \leq i \leq 3} (Q_{n_2}^{\otimes 5})^{\bar{\omega}_{(i)}^{>0}} \right) \cong (Q_{n_2}^{\otimes 5})^0 \bigoplus (\text{Ker}[\widetilde{Sq}_*^0]_{n_2} \cap (Q_{n_2}^{\otimes 5})^{>0}),$$

where $\bar{\omega}_{(1)} = (3, 2, 1, 1, 1, 1)$, $\bar{\omega}_{(2)} = (3, 4, 2, 2, 2)$ and $\bar{\omega}_{(3)} = (3, 4, 4, 3, 1)$ are the weight vectors of degree n_2 . Applying the formula $\dim(Q_{n_2}^{\otimes 5})^0 = \sum_{1 \leq s \leq 4} \binom{5}{s} \dim(Q_{n_2}^{\otimes s})^{>0}$ and the previous results by Peterson [20], Kameko [13], Sum [34], we easily obtain the dimension of $(Q_{n_2}^{\otimes 5})^0$. Thus, together with this and the above data, the dimension of $Q_{n_d}^{\otimes 5}$, $d \geq 2$, can be determined by calculating explicitly the monomial basis of $(Q_{n_2}^{\otimes 5})^{\bar{\omega}_{(i)}^{>0}}$ for $i = 1, 2, 3$. Using techniques mentioned and some preliminary calculations, we notice that determining $(Q_{n_2}^{\otimes 5})^{\bar{\omega}_{(2)}^{>0}}$ and $(Q_{n_2}^{\otimes 5})^{\bar{\omega}_{(3)}^{>0}}$ is simple, but it is not easy for $(Q_{n_2}^{\otimes 5})^{\bar{\omega}_{(1)}^{>0}}$. Ofcourse, the results can be obtained by using an algorithm in MAGMA. Additionally, a useful observation that $\mu(n_2) = 3 = \alpha(n_2 + \mu(n_2))$, and so according to Theorem 2.7 (see Sect.2), it follows that

$$\dim Q_{5(2^{d+3}-1)+n_2 \cdot 2^{d+3}}^{\otimes 6} = (2^6 - 1) \dim Q_{n_d}^{\otimes 5}, \text{ for arbitrary } d > 1.$$

Regarding to Singer's conjecture for the rank 5 case and the internal degree n_d , we have the following prediction, which is motivated by these discussions and Theorem 1.13.

Conjecture 1.15. For each non-negative integer d ,

$$\dim[\text{Ker}[\widetilde{Sq}_*^0]_{n_d}]^{GL_5} = \begin{cases} 0 & \text{if } d = 0, \\ 2 & \text{if } d = 1, \\ 1 & \text{otherwise,} \end{cases}$$

where $\text{Ker}[\widetilde{Sq}_*^0]_{n_d}$ is the kernel of Kameko's homomorphism $[\widetilde{Sq}_*^0]_{n_d} := \widetilde{Sq}_*^0 : Q_{n_d}^{\otimes 5} \rightarrow Q_{n_{d-1}}^{\otimes 5}$.

On the other side, using the results by Lin [15] and Chen [7], one has that

$$\text{Ext}_{\mathcal{A}_2}^{5,5+n_d}(\mathbb{Z}_2, \mathbb{Z}_2) = \begin{cases} \langle h_1 h_2^4, h_0^2 h_2^2 h_3 \rangle = 0 & \text{if } d = 0, \\ \langle h_0^4 h_5, n_0 \rangle & \text{if } d = 1, \\ \langle n_{d-1}, Q_3(d-2) \rangle & \text{if } d > 1. \end{cases}$$

Let us recall that the $\mathbb{Z}_2$ -cohomology of Steenrod ring $\text{Ext}_{\mathcal{A}_2}^{*,*}(\mathbb{Z}_2, \mathbb{Z}_2)$ is a bigraded algebra. Following Chen [7], let $\text{Ext}_{\mathcal{A}_2}^{t,*}(\mathbb{Z}_2, \mathbb{Z}_2) \otimes \text{Ext}_{\mathcal{A}_2}^{t',*}(\mathbb{Z}_2, \mathbb{Z}_2) \xrightarrow{\mu} \text{Ext}_{\mathcal{A}_2}^{t+t',*}(\mathbb{Z}_2, \mathbb{Z}_2)$ be the multiplication map of $\text{Ext}_{\mathcal{A}_2}^{*,*}(\mathbb{Z}_2, \mathbb{Z}_2)$. An element $\zeta \in \text{Ext}_{\mathcal{A}_2}^{\bar{t},*}(\mathbb{Z}_2, \mathbb{Z}_2)$ is said to be a *decomposable element* if $\bar{t} \geq 2$ and there exist $t > 0$, $t' > 0$ such that $t + t' = \bar{t}$ and $\zeta = \sum_j \mu(\alpha_j \otimes \beta_j)$ for some $\alpha_j \in \text{Ext}_{\mathcal{A}_2}^{t,*}(\mathbb{Z}_2, \mathbb{Z}_2)$ and $\beta_j \in \text{Ext}_{\mathcal{A}_2}^{t',*}(\mathbb{Z}_2, \mathbb{Z}_2)$. For instance, $h_0^4 h_5$ is a decomposable element in $\text{Ext}_{\mathcal{A}_2}^{5,36}(\mathbb{Z}_2, \mathbb{Z}_2)$. A non-zero element in $\text{Ext}_{\mathcal{A}_2}^{t,*}(\mathbb{Z}_2, \mathbb{Z}_2)$ is an *indecomposable element* if either $t = 1$ or if $t \geq 2$ then its image in $\text{Ext}_{\mathcal{A}_2}^{t,*}(\mathbb{Z}_2, \mathbb{Z}_2)/D^{t,*}$ is non-zero. Here the $\mathbb{Z}_2$ -submodule $D^{t,*}$ is the set of all the decomposable elements in $\text{Ext}_{\mathcal{A}_2}^{t,*}(\mathbb{Z}_2, \mathbb{Z}_2)$, where $D^{1,*} = 0$. When $t = 5$, Chen showed in [7] that $\text{Ext}_{\mathcal{A}_2}^{5,*}(\mathbb{Z}_2, \mathbb{Z}_2)$ contains eleven Sq^0 -families of indecomposable elements, namely n_d , χ_d , $D_1(d)$, $H_1(d)$, $Q_3(d)$, K_d , J_d , T_d , V_d , V'_d , and U_d , for $d \geq 0$. According to Chơn, and Hà [8], the image of $Tr_5^{\mathcal{A}_2}$ contains every element in the family $\{n_{d-1} = (Sq^0)^{d-1}(n_0)\}_{d \geq 1}$. In [12], Hưng conjectured that $Q_3(0)$ is detected by $Tr_5^{\mathcal{A}_2}$. If this prediction is true, then the image of the fifth transfer contains all the indecomposable elements of the family $\{Q_3(d-2) = (Sq^0)^{d-2}(Q_3(0))\}_{d \geq 2}$ since the following diagram is commutative:

$$\begin{array}{ccc} [P_{\mathcal{A}_2}((\mathbb{P}_{n_{d-2}}^{\otimes 5})^*)]_{GL_5} & \xrightarrow{Tr_5^{\mathcal{A}_2}} & \text{Ext}_{\mathcal{A}_2}^{5,5+n_{d-2}}(\mathbb{Z}_2, \mathbb{Z}_2) \\ \downarrow \widetilde{Sq}^0 & & \downarrow Sq^0 \\ [P_{\mathcal{A}_2}((\mathbb{P}_{5+2n_{d-2}}^{\otimes 5})^*)]_{GL_5} & \xrightarrow{Tr_5^{\mathcal{A}_2}} & \text{Ext}_{\mathcal{A}_2}^{5,2(5+n_{d-2})}(\mathbb{Z}_2, \mathbb{Z}_2). \end{array}$$

Here $\widetilde{Sq}^0$ denotes Kameko's squaring operation, while Sq^0 is the classical squaring operation. At the same time, combining with the calculations by Singer [30], Boardman [3], Bruner [5], Hà [9], and the fact that $Tr_5^{\mathcal{A}_2}$ is an algebraic homomorphism, it may be concluded that the following non-zero elements are detected by the algebraic transfer:

$$\begin{array}{ll} h_0 Q_3(0) \in \text{Ext}_{\mathcal{A}_2}^{6,6+n_2}(\mathbb{Z}_2, \mathbb{Z}_2), & h_0^2 Q_3(0) \in \text{Ext}_{\mathcal{A}_2}^{7,7+n_2}(\mathbb{Z}_2, \mathbb{Z}_2), \\ h_1 Q_3(0) \in \text{Ext}_{\mathcal{A}_2}^{6,6+n_2+1}(\mathbb{Z}_2, \mathbb{Z}_2), & h_1^2 Q_3(0) \in \text{Ext}_{\mathcal{A}_2}^{7,7+n_2+2}(\mathbb{Z}_2, \mathbb{Z}_2), \\ h_2 Q_3(0) \in \text{Ext}_{\mathcal{A}_2}^{6,6+n_2+3}(\mathbb{Z}_2, \mathbb{Z}_2), & h_0 h_2 Q_3(0) \in \text{Ext}_{\mathcal{A}_2}^{7,7+n_2+3}(\mathbb{Z}_2, \mathbb{Z}_2), \\ h_2^2 Q_3(0) \in \text{Ext}_{\mathcal{A}_2}^{7,7+n_2+6}(\mathbb{Z}_2, \mathbb{Z}_2), & c_0 Q_3(0) \in \text{Ext}_{\mathcal{A}_2}^{8,8+n_2+8}(\mathbb{Z}_2, \mathbb{Z}_2), \\ h_1 c_0 Q_3(0) \in \text{Ext}_{\mathcal{A}_2}^{9,9+n_2+9}(\mathbb{Z}_2, \mathbb{Z}_2), & d_0 Q_3(0) \in \text{Ext}_{\mathcal{A}_2}^{9,9+n_2+14}(\mathbb{Z}_2, \mathbb{Z}_2), \\ h_0 d_0 Q_3(0) \in \text{Ext}_{\mathcal{A}_2}^{10,10+n_2+14}(\mathbb{Z}_2, \mathbb{Z}_2), & h_4 Q_3(0) \in \text{Ext}_{\mathcal{A}_2}^{6,6+n_2+15}(\mathbb{Z}_2, \mathbb{Z}_2), \end{array}$$

$$\begin{aligned}
h_0 h_4 Q_3(0) &\in \text{Ext}_{\mathcal{A}_2}^{7,7+n_2+15}(\mathbb{Z}_2, \mathbb{Z}_2), & h_0^2 h_4 Q_3(0) &\in \text{Ext}_{\mathcal{A}_2}^{8,8+n_2+15}(\mathbb{Z}_2, \mathbb{Z}_2), \\
h_1 h_4 Q_3(0) &\in \text{Ext}_{\mathcal{A}_2}^{7,7+n_2+16}(\mathbb{Z}_2, \mathbb{Z}_2), & h_1^2 h_4 Q_3(0) &\in \text{Ext}_{\mathcal{A}_2}^{8,8+n_2+17}(\mathbb{Z}_2, \mathbb{Z}_2), \\
e_0 Q_3(0) &\in \text{Ext}_{\mathcal{A}_2}^{9,9+n_2+17}(\mathbb{Z}_2, \mathbb{Z}_2), & h_2 h_4 Q_3(0) &\in \text{Ext}_{\mathcal{A}_2}^{7,7+n_2+18}(\mathbb{Z}_2, \mathbb{Z}_2), \\
h_0 h_2 h_4 Q_3(0) &\in \text{Ext}_{\mathcal{A}_2}^{8,8+n_2+18}(\mathbb{Z}_2, \mathbb{Z}_2), & h_5 Q_3(0) &\in \text{Ext}_{\mathcal{A}_2}^{6,6+n_2+31}(\mathbb{Z}_2, \mathbb{Z}_2), \\
h_2 h_5 Q_3(0) &\in \text{Ext}_{\mathcal{A}_2}^{7,7+n_2+34}(\mathbb{Z}_2, \mathbb{Z}_2).
\end{aligned}$$

The above data together with Conjecture 1.15 tempt us to propose:

Conjecture 1.16. *Conjecture 1.15 for the cases $d \geq 2$ is equivalent to the following: the invariant $[(Q_{n_2}^{\otimes 5})^{\overline{\omega}(i)}]^{GL_5}$ is 1-dimensional if $i = 1$ and is trivial otherwise. Consequently, the coinvariant $[P_{\mathcal{A}_2}((\mathbb{P}_{n_d}^{\otimes 5})^*)]_{GL_5}$ is trivial if $d = 0$, and is two-dimensional otherwise. This means that Singer's transfer is an isomorphism in bidegree $(5, 5 + n_d)$ for arbitrary $d \geq 0$.*

Theorem 1.13 together with our previous work in [24] show that Conjectures 1.15 and 1.16 hold true for $d = 0, 1$. By this and the above discussions, we only have to verify these conjectures for $d = 2$. However, computationally this case seems rather non-trivial and we hope to come back to it in another work.

Summarizing, although our work does not apparently lead to a general rule for determining the space (1.2) and verifying Singer's conjecture in general rank, we feel that it represents an interesting note about an efficient approach to understand the structure of those spaces and their applications. Perhaps a continuation of our methods may prove fruitful. It is our belief that further research can spring from these ideas.

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2. Proofs of main results

To facilitate this, we will start in this section by recalling a few useful preliminaries on strictly inadmissible and spike monomials, and reminding the reader of something important.

Definition 2.1 (Strictly inadmissible monomial). A monomial $X \in \mathbb{P}_n^{\otimes t}$ is said to be *strictly inadmissible* if and only if there exist monomials $Y_1, Y_2, \dots, Y_k$ in $\mathbb{P}_n^{\otimes t}$ such that $Y_j < X$ for $1 \leq j \leq k$ and $X = \sum_{1 \leq j \leq k} Y_j + \sum_{1 \leq l \leq 2^s - 1} Sq^l(Z_l)$, where $s = \max\{i \in \mathbb{Z} : \omega_i(X) > 0\}$ and suitable polynomials $Z_l \in \mathbb{P}_{n-l}^{\otimes t}$, $n > l$.

It would be helpful for readers to note that due to Definitions 1.4 and 2.1, each strictly inadmissible monomial is inadmissible. Conversely, it is, however in general, not true; for instance, $X = x_1 x_2^2 x_3^2 x_4^2 x_5^2 x_6 \dots x_t \in \mathbb{P}_{t+4}^{\otimes t}$ is an inadmissible monomial, but it is not strictly inadmissible.

Definition 2.2 (Spike monomial). A monomial $Z = \prod_{1 \leq j \leq t} x_j^{u_j}$ in $\mathbb{P}_n^{\otimes t}$ is called a *spike* if every exponent u_j is of the form $2^{\lambda_j} - 1$. Specifically, if the exponents λ_j can be arranged to satisfy $\lambda_1 > \lambda_2 > \dots > \lambda_{s-1} \geq \lambda_s \geq 1$, where only the last two smallest exponents can be equal, and $\lambda_j = 0$ for $s < j \leq t$, then Z is called a *minimal spike*.

The following technical theorems are very important throughout this paper.

Theorem 2.3 (see Kameko [13], Sum [34]). *The following assertions are true:*

- i) Let X, Y be monomials in $\mathbb{P}^{\otimes t}$ and let s be a positive integer. Assume that there is an index $j > s$ such that $\omega_j(X) = 0$. Then, if Y is an inadmissible monomial, so is XY^{2^s} (see [13]);
- ii) Let Z, W be monomials in $\mathbb{P}^{\otimes t}$ and let r be a positive integer. Suppose that there is an index $j > r$ such that $\omega_j(Z) = 0$ and $\omega_r(Z) \neq 0$. If Z is strictly inadmissible, then ZW^{2^r} is, too (see [34]).

Theorem 2.4 (see Phc-Sum [22]). *All the spikes in $\mathbb{P}_n^{\otimes t}$ are admissible and their weight vectors are weakly decreasing. Furthermore, if a weight vector $\omega = (\omega_1, \omega_2, \dots)$ of degree n is weakly decreasing and $\omega_1 \leq t$, then there is a spike Z in $\mathbb{P}_n^{\otimes t}$ such that $\omega(Z) = \omega$.*

Theorem 2.5 (see Singer [31]). *Suppose that $X \in \mathbb{P}_n^{\otimes t}$ is a monomial of degree n , where $\mu(n) \leq t$. Let Z be the minimal spike of degree n in $\mathbb{P}_n^{\otimes t}$. If $\omega(X) < \omega(Z)$, then $X \equiv 0$.*

Theorem 2.6 (see Mothebe-Uys [17]). *Let l, d be positive integers such that $1 \leq l \leq t$. If X belongs to $\mathcal{C}_n^{\otimes(t-1)}$, then $x_l^{2^d-1}q_{(l,t)}(X)$ belongs to $\mathcal{C}_{n+2^d-1}^{\otimes t}$.*

Theorem 2.7 (see Phc [27]). *Consider generic degree of the form (1.3) with $r = t - 1$ and d, m positive integers. Suppose there is a integer ζ such that $0 \leq \zeta < d$ and $1 \leq t - 3 \leq \mu(n_\zeta) = \alpha(n_\zeta + \mu(n_\zeta)) \leq t - 2$. Then, for each $d \geq \zeta$, we have*

$$\dim Q_{(t-1)(2^d-\zeta+t-1-1)+n_\zeta 2^d-\zeta+t-1}^{\otimes t} = (2^t - 1) \dim Q_{n_d}^{\otimes(t-1)}.$$

In what follows, for a monomial $F \in \mathbb{P}_n^{\otimes t}$, we denote by $[F]$ the equivalence class of F in $Q_n^{\otimes t}$. If ω is a weight vector of degree n and $F \in (\mathbb{P}_n^{\otimes t})^{\leq \omega}$, then denote by $[F]_\omega$ the equivalence class of F in $(Q_n^{\otimes t})^\omega$. For a subset $\mathcal{C} \subset \mathbb{P}_n^{\otimes t}$, we will often write $|\mathcal{C}|$ for the cardinality of $\mathcal{C}$ and put $[\mathcal{C}] = \{[F] : F \in \mathcal{C}\}$. If $\mathcal{C} \subset (\mathbb{P}_n^{\otimes t})^{\leq \omega}$, then denote $[\mathcal{C}]_\omega = \{[F]_\omega : F \in \mathcal{C}\}$. Let us consider the sets $(\mathcal{C}_n^{\otimes t})^{\omega^0} := (\mathcal{C}_n^{\otimes t})^\omega \cap (\mathbb{P}_n^{\otimes t})^0$, and $(\mathcal{C}_n^{\otimes t})^{\omega^{>0}} := (\mathcal{C}_n^{\otimes t})^\omega \cap (\mathbb{P}_n^{\otimes t})^{>0}$, where $(\mathcal{C}_n^{\otimes t})^\omega := \mathcal{C}_n^{\otimes t} \cap (\mathbb{P}_n^{\otimes t})^{\leq \omega}$. Then, one has that the sets $[(\mathcal{C}_n^{\otimes t})^\omega]_\omega$, $[(\mathcal{C}_n^{\otimes t})^{\omega^0}]_\omega$ and $[(\mathcal{C}_n^{\otimes t})^{\omega^{>0}}]_\omega$ are respectively the bases of the $\mathbb{Z}_2$ -vector spaces $(Q_n^{\otimes t})^\omega$, $(Q_n^{\otimes t})^{\omega^0}$ and $(Q_n^{\otimes t})^{\omega^{>0}}$.

2.1. Proof of Theorem 1.6

It is known that for $t = 5$, the Kameko $\widetilde{Sq}_*^0$, which is an epimorphism, determined as follows:

$$\begin{aligned} \widetilde{Sq}_*^0 : Q_{31}^{\otimes 5} &\longrightarrow Q_{13}^{\otimes 5} \\ [\prod_{1 \leq j \leq 5} x_j^{a_j}] &\longmapsto \begin{cases} [\prod_{1 \leq j \leq 5} x_j^{\frac{a_j-1}{2}}] & \text{if } a_j \text{ odd, } j = 1, 2, \dots, 5, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

So, in order to prove theorem, we first show that if $X \in \mathcal{C}_{31}^{\otimes 5}$ such that $[X] \in \text{Ker}(\widetilde{Sq}_*^0)$, then $\omega(X)$ is one of the sequences $\omega_{(j)}$, for $1 \leq j \leq 7$, where $\omega_{(4)} := (1, 1, 1, 3)$, $\omega_{(5)} := (1, 3, 2, 2)$, $\omega_{(6)} := (1, 3, 4, 1)$ and $\omega_{(7)} := (3, 2, 4, 1)$. Indeed, we observe that $Z = x_1^{31}$ is the minimal spike in $\mathbb{P}_{31}^{\otimes 5}$ and $\omega(Z) = \omega_{(1)}$. Since $X \in \mathcal{C}_{31}^{\otimes 5}$ and $\deg(X) = 31$ odd, by Theorem 2.5, it may be concluded that $\omega_1(X) \in \{1, 3, 5\}$.

If $\omega_1(X) = 5$, then $X = \prod_{1 \leq j \leq 5} x_j Y^2$ with Y a monomial of degree 13 in $\mathbb{P}^{\otimes 5}$. Since $X \in \mathcal{C}_{31}^{\otimes 5}$ and $\omega_i(\prod_{1 \leq j \leq 5} x_j) = 0$, for all $i > 1$, according to Theorem 2.3, $Y \in \mathcal{C}_{13}^{\otimes 5}$. This implies that $\widetilde{Sq}_*^0([X]) = [Y] \neq 0$, which contradicts the fact that $[X] \in \text{Ker}(\widetilde{Sq}_*^0)$. Hence, we get either $\omega_1(X) = 1$ or $\omega_1(X) = 3$.

If $\omega_1(X) = 1$, then following Theorem 2.3, one has that $X = x_j Z^2$ with $Z \in \mathcal{C}_{15}^{\otimes 5}$ and $1 \leq j \leq 5$. By Sum [33], we get $\omega(X) \in \{\omega_{(1)}, \omega_{(4)}, \omega_{(5)}, \omega_{(6)}\}$.

If $\omega_1(X) = 3$, then by Theorem 2.3, $X = x_a x_b x_c T^2$ where $T \in \mathcal{C}_{14}^{\otimes 5}$ and $1 \leq a < b < c \leq 5$. Due to Ly-Tin [14], we deduce that $\omega(X)$ is one of the sequences $\omega_{(j)}$, $j = 2, 3, 7$.

Next, we will show that if $\omega(X) \in \{\omega_{(i)}\}_{4 \leq i \leq 7}$, then $(Q_{31}^{\otimes 5})^{\omega(X)} = 0$. As shown in Sect.1, $\dim(Q_{31}^{\otimes 5})^0 = 330$, further by a simple computation, we obtain

$$(\mathcal{C}_{31}^{\otimes 5})^0 = \bigcup_{1 \leq l \leq 5} \mathbf{q}_{(l,5)}(\mathcal{C}_{31}^{\otimes 4}) = (\mathcal{C}_{31}^{\otimes 5})^{\omega_{(1)}} \cup (\mathcal{C}_{31}^{\otimes 5})^{\omega_{(2)}},$$

where $|(\mathcal{C}_{31}^{\otimes 5})^{\omega_{(1)}}| = 30$, and $|(\mathcal{C}_{31}^{\otimes 5})^{\omega_{(2)}}| = 300$. So, $\dim(Q_{31}^{\otimes 5})^{\omega_{(1)}} = 30$, and $\dim(Q_{31}^{\otimes 5})^{\omega_{(2)}} = 300$. Thus, we need only to prove that the spaces $(Q_{31}^{\otimes 5})^{\omega_{(i)}^0}$ are trivial for $4 \leq i \leq 7$.

Case $\omega(X) = \omega_{(4)}$. It is easy to see that the monomial $X = x_j Z^2 \in (\mathbb{P}_{31}^{\otimes 5})^{>0}$, $1 \leq j \leq 5$, is one of the following monomials: $x_1^2 x_2^4 x_3^8 x_4^8 x_5^9$, $x_1^2 x_2^4 x_3^8 x_4^9 x_5^8$, $x_1^2 x_2^4 x_3^9 x_4^8 x_5^8$, $x_1^2 x_2^5 x_3^8 x_4^8 x_5^8$, $x_1^3 x_2^4 x_3^8 x_4^8 x_5^8$. They are inadmissible. Indeed, by a simple composition, we have

$$X = x_j Z^2 = x_1^3 x_2^4 x_3^8 x_4^8 x_5^8 = Sq^1(x_1^3 x_2^3 x_3^8 x_4^8 x_5^8) + Sq^2(x_1^2 x_2^3 x_3^8 x_4^8 x_5^8) + x_1^2 x_2^5 x_3^8 x_4^8 x_5^8 \mod (\mathbb{P}_{31}^{\otimes 5})^{<\omega_{(4)}}.$$

Since $x_1^2 x_2^5 x_3^8 x_4^8 x_5^8 < x_1^3 x_2^4 x_3^8 x_4^8 x_5^8$, X is inadmissible.

Case $\omega(X) = \omega_{(5)}$. Since $X = x_j Z^2$ with $Z \in \mathcal{C}_{15}^{\otimes 5}$, and $1 \leq j \leq 5$, X is a permutation of one of the following monomials:

$$\begin{array}{cccc} x_i x_j^2 x_k^2 x_l^{12} x_m^{14}, & x_i x_j^2 x_k^4 x_l^{10} x_m^{14}, & x_i x_j^2 x_k^6 x_l^8 x_m^{14}, & x_i x_j^2 x_k^6 x_l^{10} x_m^{12}, \\ x_i x_j^4 x_k^6 x_l^{10} x_m^{10}, & x_i x_j^6 x_k^6 x_l^8 x_m^{10}, & x_i^3 x_j^2 x_k^2 x_l^{12} x_m^{12}, & x_i^3 x_j^2 x_k^4 x_l^8 x_m^{14}, \\ x_i^3 x_j^2 x_k^4 x_l^{10} x_m^{12}, & x_i^3 x_j^2 x_k^6 x_l^8 x_m^{12}, & x_i^3 x_j^4 x_k^4 x_l^{10} x_m^{10}, & x_i^3 x_j^4 x_k^6 x_l^8 x_m^{10}, \\ x_i^3 x_j^6 x_k^6 x_l^8 x_m^8, & x_i^7 x_j^2 x_k^2 x_l^8 x_m^{12}, & x_i^7 x_j^2 x_k^4 x_l^8 x_m^{10}, & x_i^7 x_j^2 x_k^6 x_l^8 x_m^8, \\ x_i^{15} x_j^2 x_k^2 x_l^4 x_m^8, & & & \end{array}$$

where (i, j, k, l, m) is a permutation of $(1, 2, 3, 4, 5)$. A direct computation shows that

$$X = x_i x_j^2 x_k^2 x_l^{12} x_m^{14} = Sq^1(x_i^2 x_j x_k x_l^{12} x_m^{14}) + Sq^2(x_i x_j x_k x_l^{12} x_m^{14} + x_i x_j x_k x_l^{10} x_m^{16}) \mod (\mathbb{P}_{31}^{\otimes 5})^{<\omega_{(5)}}.$$

This implies that X is inadmissible.

Case $\omega(X) = \omega_{(6)}$. We have $X = x_j Z^2$ with $\omega(Z) = (3, 4, 1)$ and $1 \leq j \leq 5$. By a simple computation, we find that X of the form FG^{2^s} , where s is a suitable integer and F is one of the following inadmissible monomials:

$$\begin{array}{lllll} F_1 = x_1^2 x_4^2 x_5^3, & F_2 = x_1^2 x_3^2 x_5^2, & F_3 = x_1^2 x_3^2 x_5^3, & F_4 = x_1^2 x_3^2 x_4^3, & F_5 = x_1^2 x_3^3 x_5^2, \\ F_6 = x_1^2 x_3^3 x_4^2, & F_7 = x_1^2 x_2^2 x_5^3, & F_8 = x_1^2 x_2^2 x_4^3, & F_9 = x_1^2 x_2^2 x_3^3, & F_{10} = x_1^2 x_2^3 x_5^2, \\ F_{11} = x_1^2 x_2^3 x_4^2, & F_{12} = x_1^2 x_2^3 x_3^2, & F_{13} = x_1^3 x_4^2 x_5^2, & F_{14} = x_1^3 x_3^2 x_5^2, & F_{15} = x_1^3 x_3^2 x_4^2, \\ F_{16} = x_1^3 x_2^2 x_5^2, & F_{17} = x_1^3 x_2^2 x_4^2, & F_{18} = x_1^3 x_2^2 x_3^2, & F_{19} = x_1^2 x_3^2 x_4^2 x_5^2, & F_{20} = x_1^2 x_3^2 x_4^2 x_5^2, \\ F_{21} = x_1^2 x_3^2 x_4^2 x_5^2, & F_{22} = x_1^2 x_2^2 x_4^2 x_5^2, & F_{23} = x_1^2 x_2^2 x_3^2 x_5^2, & F_{24} = x_1^2 x_2^2 x_3^2 x_4^2, & F_{25} = x_1^2 x_2^2 x_4^2 x_5^2, \\ F_{26} = x_1^2 x_2^2 x_4^2 x_5^2, & F_{27} = x_1^2 x_2^2 x_3^2 x_5^2, & F_{28} = x_1^2 x_2^2 x_3^2 x_4^2, & F_{29} = x_1^2 x_2^2 x_3^2 x_5^2, & F_{30} = x_1^2 x_2^2 x_3^2 x_4^2. \end{array}$$

Then, according to Theorem 2.3, X is inadmissible.

Case $\omega(X) = \omega_{(7)}$. Note that $X = x_a x_b x_c T^2$ with $T \in \mathcal{C}_{14}^{\otimes 5}$, and $1 \leq a < b < c \leq 5$. Then, X is a permutation of one of the following monomials:

$$\begin{array}{cccc} x_i^3 x_j^4 x_k^4 x_l^5 x_m^{15}, & x_i^3 x_j^4 x_k^4 x_l^7 x_m^{13}, & x_i^3 x_j^4 x_k^5 x_l^5 x_m^{14}, & x_i^3 x_j^4 x_k^5 x_l^6 x_m^{13}, \\ x_i^3 x_j^4 x_k^5 x_l^7 x_m^{12}, & x_i^3 x_j^5 x_k^5 x_l^6 x_m^{12}, & x_i^7 x_j^2 x_k^4 x_l^5 x_m^{13}, & x_i^7 x_j^2 x_k^5 x_l^5 x_m^{12}, \\ x_i^7 x_j^4 x_k^4 x_l^5 x_m^{11}, & x_i^7 x_j^4 x_k^5 x_l^5 x_m^{10}, & x_i^{15} x_j^2 x_k^4 x_l^5 x_m^5, & \end{array}$$

where (i, j, k, l, m) is a permutation of $(1, 2, 3, 4, 5)$. Using the Cartan formula, one has

$$\begin{aligned} X = x_i^3 x_j^4 x_k^4 x_l^5 x_m^{15} &= Sq^1(x_i^3 x_j x_k^2 x_l^9 x_m^{15} + x_i^3 x_j x_k^2 x_l^5 x_m^{19}) \\ &+ Sq^2(x_i^5 x_j^2 x_k^2 x_l^5 x_m^{15} + x_i^5 x_j x_k^2 x_l^6 x_m^{15}) \\ &+ Sq^4(x_i^3 x_j^2 x_k^2 x_l^5 x_m^{15}) \mod (\mathbb{P}_{31}^{\otimes 5})^{<\omega_{(7)}}, \end{aligned}$$

and we therefore deduce that X is inadmissible.

Thus the above computations show that $(Q_{31}^{\otimes 5})^{\omega_{(j)}^{>0}} = 0$ for $j \in \{4, 5, 6, 7\}$. Now, from the above cases, we have a direct summand decomposition of the $\mathbb{Z}_2$ -vector spaces:

$$\text{Ker}(\widetilde{Sq}_*^0) \cap (Q_{31}^{\otimes 5})^{>0} = (Q_{31}^{\otimes 5})^{\omega_{(1)}^{>0}} \oplus (Q_{31}^{\otimes 5})^{\omega_{(2)}^{>0}} \oplus (Q_{31}^{\otimes 5})^{\omega_{(3)}^{>0}},$$

from which we deduce that

$$\begin{aligned} \text{Ker}(\widetilde{Sq}_*^0) &\cong (Q_{31}^{\otimes 5})^0 \oplus (\text{Ker}(\widetilde{Sq}_*^0) \cap (Q_{31}^{\otimes 5})^{>0}) \\ &= (Q_{31}^{\otimes 5})^0 \oplus \left(\bigoplus_{1 \leq j \leq 3} (Q_{31}^{\otimes 5})^{\omega_{(j)}^{>0}} \right). \end{aligned}$$

This finishes the proof of Theorem 1.6.

2.2. Proof of Theorem 1.7

In this section, we use the following homomorphisms: For any $(l, \mathcal{L}) \in \mathcal{N}_5$, the $\mathbb{Z}_2$ -linear transformation $\mathbf{p}_{(l, \mathcal{L})} : \mathbb{P}_n^{\otimes 5} \rightarrow \mathbb{P}_n^{\otimes 4}$ defined by

$$\mathbf{p}_{(l, \mathcal{L})}(x_j) = \begin{cases} x_j & \text{if } 1 \leq j \leq l-1, \\ \sum_{p \in \mathcal{L}} x_{p-1} & \text{if } j = l, \\ x_{j-1} & \text{if } l+1 \leq j \leq 5. \end{cases}$$

It is remarkable that $\mathbf{p}_{(l, \emptyset)}(x_l) = 0$ for all l , $1 \leq l \leq 5$. Furthermore, this linear map induces a homomorphism of $\mathcal{A}_2$ -algebras which is also denoted by $\mathbf{p}_{(l, \mathcal{L})} : \mathbb{P}_n^{\otimes 5} \rightarrow \mathbb{P}_n^{\otimes 4}$. In particular, if X is a monomial in $\mathbb{P}_n^{\otimes 5}$, then $\mathbf{p}_{(l, \mathcal{L})}(X) \in (\mathbb{P}_n^{\otimes 4})^{\leq \omega(X)}$ (see [22]). Now, the proof of this theorem follows from the statements below.

Proposition 2.2.1. *We have $(Q_{31}^{\otimes 5})^{\omega_{(1)}^{>0}} = \langle [x_1 x_2^2 x_3^4 x_4^8 x_5^{16}] \rangle$, and $[x_1 x_2^2 x_3^4 x_4^8 x_5^{16}] \neq 0$.*

Proof. Let $X \in (\mathcal{C}_{31}^{\otimes 5})^{\omega_{(1)}^{>0}}$. Since $X = x_j Z^2$ with $Z \in (\mathcal{C}_{15}^{\otimes 5})^{>0}$, $1 \leq j \leq 5$, by a simple computation, we see that if $X \neq x_1 x_2^2 x_3^4 x_4^8 x_5^{16}$, then X is a permutation of the following monomials:

$$\begin{array}{cccc} x_i^2 x_j x_k^4 x_l^8 x_m^{16}, & x_l^2 x_m^{29}, & x_l^3 x_m^{28}, & x_l x_m^{30}, \\ x_k x_l^2 x_m^{28}, & x_k^2 x_l^4 x_m^{25}, & x_k^2 x_l^5 x_m^{24}, & x_k^3 x_l^4 x_m^{24}, \\ x_j x_k^2 x_l^4 x_m^{24}, & x_j^2 x_k^4 x_l^8 x_m^{17}, & x_j^2 x_k^4 x_l^9 x_m^{16}, & x_j^2 x_k^5 x_l^8 x_m^{16}, \\ x_j^3 x_k^4 x_l^8 x_m^{16}, & & & \end{array}$$

where i, j, k, l, m are distinct integers and $1 \leq i, j, k, l, m \leq 5$. It is easy to see that

$$X = x_i^2 x_j x_k^4 x_l^8 x_m^{16} = Sq^1(x_i x_j x_k^4 x_l^8 x_m^{16}) + x_i x_j^2 x_k^4 x_l^8 x_m^{16}$$

and $x_i x_j^2 x_k^4 x_l^8 x_m^{16} < x_i^2 x_j x_k^4 x_l^8 x_m^{16}$. Hence, X is inadmissible. Now suppose there exist a linear relation $\gamma x_1 x_2^2 x_3^4 x_4^8 x_5^{16} \equiv 0$ with $\gamma \in \mathbb{Z}_2$. Then the homomorphism $\mathbf{p}_{(4, (5))} : \mathbb{P}^{\otimes 5} \rightarrow \mathbb{P}^{\otimes 4}$ sends this equality to $\mathbf{p}_{(4, (5))}(\gamma x_1 x_2^2 x_3^4 x_4^8 x_5^{16}) \equiv \gamma x_1 x_2^2 x_3^4 x_4^{24} \equiv 0$ and so $\gamma = 0$ since according to Sum [34], $x_1 x_2^2 x_3^4 x_4^{24}$ is an admissible monomial in $\mathbb{P}_{31}^{\otimes 4}$. The proposition follows. $\square$

The following (Lemmata 2.2.2 and 2.2.3) are crucial observations in the proof of the theorem.

Lemma 2.2.2. *The following monomials are strictly inadmissible:*

$$\begin{array}{cccc} x_1 x_2^{14} x_3^3 x_4^{12} x_5, & x_1^3 x_2^{12} x_3 x_4 x_5^{14}, & x_1^3 x_2^{12} x_3 x_4^{14} x_5, & x_1 x_2^6 x_3^{11} x_4^{12} x_5, \\ x_1^3 x_2^{12} x_3 x_4^{13} x_5, & x_1^3 x_2^{12} x_3^8 x_4^{12} x_5, & x_1^3 x_2^{12} x_3^3 x_4^{12} x_5, & x_1^3 x_2^{12} x_3^3 x_4 x_5^{12}, \\ x_1^3 x_2^{12} x_3^3 x_4^{12} x_5, & x_1^3 x_2^{15} x_3^4 x_4^8 x_5, & x_1^3 x_2^{15} x_3^4 x_4^8 x_5, & x_1^3 x_2^4 x_3^4 x_4^{15} x_5, \\ x_1^3 x_2^4 x_3^{15} x_4^8 x_5, & x_1^3 x_2^4 x_3^8 x_4^{15} x_5, & x_1^3 x_2^4 x_3^{15} x_4^8 x_5, & x_1^3 x_2^4 x_3^8 x_4^{15} x_5, \\ x_1^3 x_2^4 x_3^8 x_4^{15} x_5, & x_1^{15} x_2^3 x_4^4 x_8^8, & x_1^{15} x_2^3 x_4^8 x_8^8, & x_1 x_2^{14} x_3^3 x_4^9 x_5, \\ x_1^3 x_2^4 x_3^9 x_4^{14} x_5, & x_1^3 x_2^4 x_3^9 x_4^{14} x_5, & x_1^3 x_2^4 x_3^9 x_4^{14} x_5, & x_1^3 x_2^4 x_3^9 x_4^{14} x_5, \\ x_1^3 x_2^4 x_3^{11} x_4^{12} x_5, & x_1 x_2^{14} x_3^3 x_4^8 x_5, & x_1^3 x_2^5 x_3^8 x_4^{14} x_5, & x_1^3 x_2^5 x_3^8 x_4^{14} x_5, \\ x_1^3 x_2^5 x_3^{14} x_4^8 x_5, & x_1^3 x_2^5 x_3^8 x_4^{14} x_5, & x_1^3 x_2^5 x_3^{14} x_4^8 x_5, & x_1^3 x_2^5 x_3^8 x_4^{14} x_5, \\ x_1 x_2^6 x_3^3 x_4^{13} x_5, & x_1 x_2^6 x_3^3 x_4^{12} x_5^9, & x_1^3 x_2^7 x_3^{12} x_4^8 x_5, & x_1^3 x_2^7 x_3^{12} x_4^8 x_5, \\ x_1 x_2^7 x_3^{10} x_4^{12} x_5, & x_1 x_2^7 x_3^{10} x_4^{12} x_5, & & \end{array}$$

Proof. Let us consider the monomials $x_1x_2^6x_3^3x_4^{12}x_5^9$, and $x_1^3x_2^7x_3^{12}x_4x_5^8$. Because the statements are similar, we will only prove the lemma for these monomials. By a direct computation using the Cartan formula, we get the equality below.

$$\begin{aligned}
x_1x_2^6x_3^3x_4^{12}x_5^9 &= x_1x_2^3x_3^5x_4^8x_5^{14} + x_1x_2^3x_3^6x_4^8x_5^{13} + x_1x_2^3x_3^6x_4^{12}x_5^9 + x_1x_2^3x_3^8x_4^5x_5^{14} \\
&\quad + x_1x_2^3x_3^8x_4^6x_5^{13} + x_1x_2^3x_3^8x_4^{12}x_5^7 + x_1x_2^4x_3^3x_4^9x_5^{14} + x_1x_2^4x_3^3x_4^{10}x_5^{13} \\
&\quad + x_1x_2^4x_3^3x_4^{12}x_5^{11} + x_1x_2^4x_3^5x_4^{10}x_5^{11} + x_1x_2^4x_3^9x_4^{10}x_5^7 + x_1x_2^4x_3^{10}x_4^5x_5^{11} \\
&\quad + x_1x_2^4x_3^{10}x_4^9x_5^7 + x_1x_2^5x_3^5x_4^{10}x_5^{10} + x_1x_2^5x_3^6x_4^9x_5^{10} + x_1x_2^5x_3^8x_4^{10}x_5^7 \\
&\quad + x_1x_2^6x_3^2x_4^9x_5^{13} + x_1x_2^6x_3^3x_4^8x_5^{13} + x_1x_2^6x_3^3x_4^9x_5^{12} \\
&\quad + Sq^1\left(x_1x_2^3x_3^5x_4^8x_5^{13} + x_1x_2^3x_3^8x_4^5x_5^{13} + x_1x_2^5x_3^5x_4^5x_5^{14} + x_1x_2^5x_3^5x_4^6x_5^{13} \right. \\
&\quad + x_1x_2^5x_3^5x_4^{10}x_5^9 + x_1x_2^5x_3^5x_4^{12}x_5^7 + x_1x_2^5x_3^6x_4^5x_5^{13} + x_1x_2^5x_3^6x_4^9x_5^9 \\
&\quad + x_1x_2^5x_3^8x_4^9x_5^7 + x_1x_2^6x_3^5x_4^9x_5^9 + x_1x_2^8x_3^5x_4^9x_5^7 + x_1^2x_2^3x_3^5x_4^5x_5^{13} \\
&\quad \left. + x_1^4x_2^5x_3^5x_4^9x_5^7\right) \\
&\quad + Sq^2\left(x_1x_2^3x_3^6x_4^{12}x_5^7 + x_1x_2^6x_3^2x_4^9x_5^{11} + x_1x_2^6x_3^3x_4^5x_5^{14} + x_1x_2^6x_3^3x_4^6x_5^{13} \right. \\
&\quad + x_1x_2^6x_3^3x_4^9x_5^{10} + x_1x_2^6x_3^3x_4^{12}x_5^7 + x_1x_2^6x_3^6x_4^5x_5^{11} + x_1^2x_2^3x_3^5x_4^5x_5^{14} \\
&\quad + x_1^2x_2^3x_3^5x_4^6x_5^{13} + x_1^2x_2^3x_3^6x_4^5x_5^{13} + x_1^2x_2^5x_3^5x_4^{10}x_5^7 + x_1^2x_2^5x_3^6x_4^9x_5^7 \\
&\quad \left. + x_1^2x_2^6x_3^5x_4^9x_5^7\right) \\
&\quad + Sq^4\left(x_1x_2^4x_3^3x_4^5x_5^{14} + x_1x_2^4x_3^3x_4^6x_5^{13} + x_1x_2^4x_3^3x_4^{12}x_5^7 + x_1x_2^4x_3^5x_4^{10}x_5^7 \right. \\
&\quad \left. + x_1x_2^4x_3^6x_4^5x_5^{11} + x_1x_2^4x_3^6x_4^9x_5^7 + x_1x_2^{10}x_3^4x_4^5x_5^7\right) + Sq^8(x_1x_2^6x_3^4x_4^5x_5^7) \pmod{(\mathbb{P}_{31}^{\otimes 5})^{<\omega(2)}}.
\end{aligned}$$

This shows that $x_1x_2^6x_3^3x_4^{12}x_5^9$ is strictly inadmissible. Next, the following computation implies that $x_1^3x_2^7x_3^{12}x_4x_5^8$ is also strictly inadmissible:

$$\begin{aligned}
x_1^3x_2^7x_3^{12}x_4x_5^8 &= x_1^2x_2^{11}x_3^5x_4x_5^{12} + x_1^2x_2^{11}x_3^5x_4^4x_5^9 + x_1^2x_2^{11}x_3^5x_4^8x_5^5 + x_1^2x_2^{13}x_3^3x_4^4x_5^9 \\
&\quad + x_1^2x_2^{13}x_3^3x_4^8x_5^5 + x_1^2x_2^{13}x_3^5x_4x_5^{10} + x_1^2x_2^{13}x_3^9x_4x_5^6 + x_1^3x_2^3x_3^5x_4^8x_5^{12} \\
&\quad + x_1^3x_2^3x_3^9x_4^4x_5^{12} + x_1^3x_2^3x_3^{12}x_4x_5^{12} + x_1^3x_2^5x_3^5x_4^8x_5^{10} + x_1^3x_2^5x_3^6x_4^8x_5^9 \\
&\quad + x_1^3x_2^5x_3^9x_4^4x_5^{10} + x_1^3x_2^5x_3^9x_4^8x_5^6 + x_1^3x_2^5x_3^{10}x_4^4x_5^9 + x_1^3x_2^5x_3^{10}x_4^8x_5^5 \\
&\quad + x_1^3x_2^7x_3^5x_4^8x_5^8 + x_1^3x_2^7x_3^8x_4x_5^{12} + x_1^3x_2^7x_3^8x_4^4x_5^9 + x_1^3x_2^7x_3^8x_4^8x_5^5 \\
&\quad + x_1^3x_2^7x_3^9x_4^4x_5^8 + Sq^1\left(x_1^5x_2^3x_3^5x_4^8x_5^9 + x_1^5x_2^3x_3^9x_4x_5^{12} + x_1^5x_2^7x_3^5x_4x_5^{12} \right. \\
&\quad \left. + x_1^5x_2^7x_3^5x_4^4x_5^9 + x_1^5x_2^7x_3^5x_4^8x_5^5 + x_1^5x_2^7x_3^9x_4x_5^8\right) \\
&\quad + Sq^2\left(x_1^2x_2^{11}x_3^3x_4^4x_5^9 + x_1^2x_2^{11}x_3^3x_4^8x_5^5 + x_1^2x_2^{11}x_3^5x_4x_5^{10} + x_1^2x_2^{11}x_3^9x_4x_5^6 \right. \\
&\quad + x_1^3x_2^3x_3^5x_4^8x_5^{10} + x_1^3x_2^3x_3^6x_4^4x_5^9 + x_1^3x_2^3x_3^9x_4^2x_5^{12} + x_1^3x_2^3x_3^{10}x_4x_5^{12} \\
&\quad + x_1^3x_2^7x_3^5x_4^2x_5^{12} + x_1^3x_2^7x_3^5x_4^4x_5^{10} + x_1^3x_2^7x_3^5x_4^8x_5^6 + x_1^3x_2^7x_3^6x_4x_5^{12} \\
&\quad + x_1^3x_2^7x_3^6x_4^4x_5^9 + x_1^3x_2^7x_3^6x_4^8x_5^5 + x_1^3x_2^7x_3^9x_4^2x_5^8 + x_1^3x_2^7x_3^{10}x_4x_5^8 \\
&\quad + x_1^6x_2^3x_3^3x_4^8x_5^9 + x_1^6x_2^3x_3^9x_4x_5^{10} + x_1^6x_2^7x_3^3x_4^4x_5^9 + x_1^6x_2^7x_3^3x_4^8x_5^5 \\
&\quad \left. + x_1^6x_2^7x_3^5x_4x_5^{10} + x_1^6x_2^7x_3^9x_4x_5^6\right)
\end{aligned}$$

$$\begin{aligned}
& +Sq^4\left(x_1^3x_2^5x_3^5x_4^2x_5^{12} + x_1^3x_2^5x_3^5x_4^4x_5^{10} + x_1^3x_2^5x_3^5x_4^8x_5^6 + x_1^3x_2^5x_3^6x_4x_5^{12}\right. \\
& + x_1^3x_2^5x_3^6x_4^4x_5^9 + x_1^3x_2^5x_3^6x_4^8x_5^5 + x_1^4x_2^7x_3^3x_4^4x_5^9 + x_1^4x_2^7x_3^3x_4^8x_5^5 \\
& + x_1^4x_2^7x_3^5x_4x_5^{10} + x_1^4x_2^7x_3^9x_4x_5^6 + x_1^{10}x_2^5x_3^3x_4^4x_5^5 + x_1^{10}x_2^5x_3^5x_4x_5^6\Big) \\
& +Sq^8\left(x_1^6x_2^5x_3^3x_4^4x_5^5 + x_1^6x_2^5x_3^5x_4x_5^6\right) \mod (\mathbb{P}_{31}^{\otimes 5})^{<\omega(2)}.
\end{aligned}$$

The lemma follows. $\square$

Lemma 2.2.3. *The following monomials are strictly inadmissible:*

$$\begin{array}{cccc}
x_1^3x_2^7x_3^{12}x_4^8x_5, & x_1^7x_2^3x_3^{12}x_4x_5^8, & x_1^7x_2^3x_3^8x_4^{12}x_5, & x_1^7x_2^3x_3^{12}x_4^8x_5, \\
x_1x_2^6x_3^{11}x_4^4x_5^9, & x_1^7x_2^{11}x_3^4x_4x_5^8, & x_1^7x_2^{11}x_3^4x_4^8x_5, & x_1x_2^6x_3^{11}x_4^5x_5^8, \\
x_1^7x_2^8x_3^8x_4^8x_5, & x_1^3x_2^4x_3^9x_4^2x_5^{13}, & x_1^3x_2^5x_3^8x_4^2x_5^{13}, & x_1^3x_2^5x_3^9x_4^2x_5^{12}, \\
x_1^3x_2^4x_3^{13}x_4^8x_5, & x_1^3x_2^4x_3^3x_4^8x_5^{13}, & x_1^3x_2^4x_3^8x_4^3x_5^{13}, & x_1^3x_2^4x_3^9x_4^3x_5^{12}, \\
x_1^3x_2^4x_3^3x_4^{12}x_5^9, & x_1^3x_2^{12}x_3^3x_4^4x_5^9, & x_1^3x_2^5x_3^8x_4^3x_5^{12}, & x_1^3x_2^{12}x_3^3x_4^5x_5^8, \\
x_1^3x_2^4x_3^{11}x_4^4x_5^9, & x_1^3x_2^4x_3^{11}x_4^5x_5^8, & x_1^3x_2^4x_3^9x_4^6x_5^9, & x_1^3x_2^7x_3^8x_4^4x_5^9, \\
x_1^3x_2^4x_3^7x_4^9x_5^8, & x_1^3x_2^4x_3^9x_4^7x_5^8, & x_1^3x_2^4x_3^7x_4^8x_5^9, & x_1^3x_2^4x_3^8x_4^7x_5^9, \\
x_1^7x_2^3x_3^8x_4^4x_5^9, & x_1^7x_2^8x_3^3x_4^4x_5^9, & x_1^3x_2^5x_3^9x_4^6x_5^8, & x_1^3x_2^5x_3^8x_4^6x_5^9, \\
x_1^3x_2^7x_3^8x_4^5x_5^8, & x_1^3x_2^5x_3^8x_4^7x_5^8, & x_1^7x_2^3x_3^8x_4^5x_5^8, & x_1^7x_2^8x_3^3x_4^5x_5^8.
\end{array}$$

Proof. We prove the lemma for the monomials $x_1^3x_2^4x_3^3x_4^8x_5^{13}$, and $x_1^7x_2^7x_3^8x_4^8x_5$. Computing these monomials is long and technical. The others can be proved by a similar technique. A direct computation shows that

$$\begin{aligned}
x_1^3x_2^4x_3^3x_4^8x_5^{13} &= Sq^1\left(x_1^3x_2x_3^3x_4^6x_5^{17} + x_1^3x_2x_3^3x_4^9x_5^{14} + x_1^3x_2x_3^3x_4^{10}x_5^{13} + x_1^3x_2x_3^6x_4^9x_5^{11}\right. \\
& + x_1^3x_2x_3^{10}x_4^5x_5^{11} + x_1^3x_2^3x_3^6x_4^6x_5^{17} + x_1^3x_2^3x_3^6x_4^{10}x_5^{13} + x_1^3x_2^3x_3^6x_4^{12}x_5^{11} \\
& + x_1^3x_2^3x_3^6x_4^{16}x_5^7 + x_1^3x_2^3x_3^4x_4^3x_5^{17} + x_1^3x_2^3x_3^8x_4^3x_5^{13} + x_1^3x_2^3x_3^8x_4^5x_5^{11} \\
& \left. + x_1^3x_2^3x_3^8x_4^9x_5^7 + x_1^3x_2^4x_3^5x_4^5x_5^{13} + x_1^5x_2^5x_3x_4^5x_5^{14}\right) \\
& + Sq^2\left(x_1^2x_2x_3^3x_4^9x_5^{14} + x_1^2x_2x_3^3x_4^{10}x_5^{13} + x_1^2x_2x_3^6x_4^9x_5^{11} + x_1^2x_2x_3^{10}x_4^5x_5^{11}\right. \\
& + x_1^2x_2^3x_3^9x_4^5x_5^{14} + x_1^2x_2^3x_3^6x_4^{10}x_5^{13} + x_1^2x_2^3x_3^6x_4^{12}x_5^{11} + x_1^2x_2^3x_3^8x_4^3x_5^{13} \\
& + x_1^2x_2^3x_3^8x_4^5x_5^{11} + x_1^2x_2^3x_3^8x_4^9x_5^7 + x_1^3x_2^5x_3^6x_4^6x_5^{14} + x_1^3x_2^5x_3^2x_4^5x_5^{14} \\
& + x_1^3x_2^6x_3^6x_4^5x_5^{14} + x_1^5x_2^2x_3^6x_4^6x_5^{11} + x_1^5x_2^2x_3^2x_4^9x_5^{11} + x_1^5x_2^2x_3^3x_4^5x_5^{14} \\
& + x_1^5x_2^2x_3^6x_4^5x_5^{13} + x_1^5x_2^2x_3^6x_4^5x_5^{11} + x_1^5x_2^2x_3^8x_4^3x_5^{11} + x_1^5x_2^3x_3^6x_4^6x_5^{14} \\
& + x_1^5x_2^3x_3^2x_4^6x_5^{13} + x_1^5x_2^3x_3^2x_4^8x_5^{11} + x_1^5x_2^3x_3^2x_4^{10}x_5^9 + x_1^5x_2^3x_3^2x_4^{12}x_5^7 \\
& + x_1^5x_2^3x_3^4x_4^3x_5^{14} + x_1^5x_2^3x_3^4x_4^6x_5^{11} + x_1^5x_2^3x_3^4x_4^{10}x_5^7 + x_1^6x_2^3x_3^5x_4^5x_5^{14}\Big) \\
& + Sq^4\left(x_1^3x_2x_3^6x_4^5x_5^{12} + x_1^3x_2x_3^6x_4^6x_5^{11} + x_1^3x_2^2x_3^2x_4^9x_5^{11} + x_1^3x_2^2x_3^3x_4^5x_5^{14}\right. \\
& + x_1^3x_2^2x_3^3x_4^6x_5^{13} + x_1^3x_2^2x_3^6x_4^5x_5^{11} + x_1^3x_2^2x_3^8x_4^3x_5^{11} + x_1^3x_2^2x_3^6x_4^6x_5^{14} \\
& + x_1^3x_2^3x_3^2x_4^6x_5^{13} + x_1^3x_2^3x_3^2x_4^8x_5^{11} + x_1^3x_2^3x_3^2x_4^{10}x_5^9 + x_1^3x_2^3x_3^2x_4^{12}x_5^7 \\
& + x_1^3x_2^3x_3^4x_4^3x_5^{14} + x_1^3x_2^3x_3^4x_4^6x_5^{11} + x_1^3x_2^3x_3^4x_4^{10}x_5^7 + x_1^3x_2^4x_3^5x_4^5x_5^{14} \\
& \left. + x_1^3x_2^8x_3^4x_4^5x_5^7 + x_1^4x_2^3x_3^5x_4^5x_5^{14}\right) + Sq^8\left(x_1^3x_2^4x_3^4x_4^5x_5^7\right)
\end{aligned}$$

$$\begin{aligned}
& +x_1^2x_2x_3^3x_4^{12}x_5^{13} + x_1^2x_2x_3^5x_4^9x_5^{14} + x_1^2x_2x_3^5x_4^{10}x_5^{13} + x_1^2x_2x_3^6x_4^9x_5^{13} \\
& +x_1^2x_2x_3^{10}x_4^5x_5^{13} + x_1^2x_2x_3^{12}x_4^5x_5^{11} + x_1^2x_2^5x_3x_4^9x_5^{14} + x_1^2x_2^5x_3x_4^{10}x_5^{13} \\
& +x_1^2x_2^5x_3x_4^{12}x_5^{11} + x_1^2x_2^5x_3^8x_4^3x_5^{13} + x_1^2x_2^5x_3^8x_4^5x_5^{11} + x_1^2x_2^5x_3^8x_4^9x_5^7 \\
& +x_1^3x_2x_3^4x_4^9x_5^{14} + x_1^3x_2x_3^4x_4^{10}x_5^{13} + x_1^3x_2x_3^6x_4^8x_5^{13} + x_1^3x_2x_3^6x_4^6x_5^{13} \\
& +x_1^3x_2^2x_3^4x_4^9x_5^{13} + x_1^3x_2^2x_3^5x_4^8x_5^{13} + x_1^3x_2^3x_3x_4^{12}x_5^{12} + x_1^3x_2^3x_3^4x_4^{12}x_5^9 \\
& +x_1^3x_2^3x_3^8x_4^4x_5^{13} + x_1^3x_2^3x_3^8x_4^5x_5^{12} + x_1^3x_2^4x_3x_4^9x_5^{14} + x_1^3x_2^4x_3x_4^{10}x_5^{13} \\
& +x_1^3x_2^4x_3x_4^{12}x_5^{11} + x_1^3x_2^4x_3^2x_4^9x_5^{13} \pmod{(\mathbb{P}_{31}^{\otimes 5})^{<\omega(2)}},
\end{aligned}$$

from which we deduce that $x_1^3x_2^4x_3^3x_4^8x_5^{13}$ is strictly inadmissible. Next, we have

$$\begin{aligned}
x_1^7x_2^7x_3^8x_4^8x_5 &= Sq^1(x_1^7x_2^{13}x_3x_4^4x_5^5 + x_1^7x_2^{13}x_3x_4^5x_5^4 + x_1^7x_2^{13}x_3^4x_4x_5^5 + x_1^7x_2^{13}x_3^4x_5^4x_5 \\
&+ x_1^7x_2^{13}x_3^5x_4x_5^4 + x_1^7x_2^{13}x_3^5x_4^4x_5) \\
&+ Sq^2(x_1^7x_2^{11}x_3x_4^4x_5^6 + x_1^7x_2^{11}x_3x_4^6x_5^4 + x_1^7x_2^{11}x_3^4x_4x_5^6 + x_1^7x_2^{11}x_3^4x_4^6x_5 \\
&+ x_1^7x_2^{11}x_3^6x_4x_5^4 + x_1^7x_2^{11}x_3^6x_4^4x_5 + x_1^7x_2^{13}x_3^2x_4^2x_5^5 + x_1^7x_2^{13}x_3^2x_5^4x_5^2 \\
&+ x_1^7x_2^{13}x_3^5x_4^2x_5^2 + x_1^7x_2^{14}x_3x_4^2x_5^5 + x_1^7x_2^{14}x_3x_4^3x_5^4 + x_1^7x_2^{14}x_3^2x_4x_5^5 \\
&+ x_1^7x_2^{14}x_3^3x_4x_5^4 + x_1^7x_2^{14}x_3^3x_4^4x_5 + x_1^7x_2^{14}x_3^4x_4^3x_5) \\
&+ Sq^4(x_1^5x_2^{11}x_3x_4^4x_5^6 + x_1^5x_2^{11}x_3x_4^6x_5^4 + x_1^5x_2^{11}x_3^4x_4x_5^6 + x_1^5x_2^{11}x_3^4x_4^6x_5 \\
&+ x_1^5x_2^{11}x_3^6x_4x_5^4 + x_1^5x_2^{11}x_3^6x_4^4x_5 + x_1^5x_2^{13}x_3^2x_4^2x_5^5 + x_1^5x_2^{13}x_3^2x_4^5x_5^2 \\
&+ x_1^5x_2^{13}x_3^5x_4^2x_5^2 + x_1^5x_2^{14}x_3x_4^2x_5^5 + x_1^5x_2^{14}x_3x_4^3x_5^4 + x_1^5x_2^{14}x_3^2x_4x_5^5 \\
&+ x_1^5x_2^{14}x_3^3x_4x_5^4 + x_1^5x_2^{14}x_3^3x_4^4x_5 + x_1^5x_2^{14}x_3^4x_4^3x_5 + x_1^{11}x_2^7x_3x_4^4x_5^4 \\
&+ x_1^{11}x_2^7x_3^4x_4x_5^4 + x_1^{11}x_2^7x_3^4x_4^4x_5) \\
&+ Sq^8(x_1^7x_2^7x_3x_4^4x_5^4 + x_1^7x_2^7x_3^4x_4x_5^4 + x_1^7x_2^7x_3^4x_4^4x_5) \\
&+ x_1^5x_2^{11}x_3x_4^4x_5^{10} + x_1^5x_2^{11}x_3x_4^6x_5^8 + x_1^5x_2^{11}x_3x_4^8x_5^6 + x_1^5x_2^{11}x_3x_4^{10}x_5^4 \\
&+ x_1^5x_2^{11}x_3^4x_4x_5^{10} + x_1^5x_2^{11}x_3^4x_4^{10}x_5 + x_1^5x_2^{11}x_3^6x_4x_5^8 + x_1^5x_2^{11}x_3^6x_4^8x_5 \\
&+ x_1^5x_2^{11}x_3^8x_4x_5^6 + x_1^5x_2^{11}x_3^8x_4^6x_5 + x_1^5x_2^{11}x_3^{10}x_4x_5^4 + x_1^5x_2^{11}x_3^{10}x_4^4x_5 \\
&+ x_1^5x_2^{13}x_3^2x_4^2x_5^9 + x_1^5x_2^{13}x_3^2x_4^9x_5^2 + x_1^5x_2^{13}x_3^9x_4^2x_5^2 + x_1^5x_2^{14}x_3x_4^2x_5^9 \\
&+ x_1^5x_2^{14}x_3x_4^3x_5^8 + x_1^5x_2^{14}x_3^2x_4x_5^9 + x_1^5x_2^{14}x_3^3x_4x_5^8 + x_1^5x_2^{14}x_3^3x_4^8x_5 \\
&+ x_1^5x_2^{14}x_3^8x_4^3x_5 + x_1^7x_2^7x_3x_4^8x_5^8 + x_1^7x_2^7x_3^8x_4x_5^8 \pmod{(\mathbb{P}_{31}^{\otimes 5})^{<\omega(2)}}.
\end{aligned}$$

These equalities show that $x_1^7x_2^7x_3^8x_4^8x_5$ is also strictly inadmissible. The lemma is proven. $\square$

Now, for each $1 \leq l \leq 5$, we put

$$\begin{aligned}
A(30) &= \{x_l \mathbf{q}_{(l,5)}(X) : X \in (\mathcal{C}_{30}^{\otimes 4})^{>0}\}, \\
A(28) &= \{x_l^3 \mathbf{q}_{(l,5)}(X) : X \in (\mathcal{C}_{28}^{\otimes 4})^{>0}\}, \\
A(24) &= \{x_l^7 \mathbf{q}_{(l,5)}(X) : X \in (\mathcal{C}_{24}^{\otimes 4})^{>0}\}, \\
A(16) &= \{x_l^{15} \mathbf{q}_{(l,5)}(X) : X \in (\mathcal{C}_{16}^{\otimes 4})^{>0}\},
\end{aligned}$$

and $A(\omega_{(i)}) = (\mathbb{P}^{\otimes 5})^{\omega_{(i)}^{>0}} \cap (A(30) \cup A(28) \cup A(24) \cup A(16))$, where

$$(\mathbb{P}^{\otimes 5})^{\omega_{(i)}^{>0}} := \langle \{X \in (\mathbb{P}^{\otimes 5})^{>0} : \omega(X) = \omega_{(i)}\} \rangle, \quad i = 2, 3.$$

It is not hard to show that $A(\omega_{(2)}) = \{X_j \mid 1 \leq j \leq 68\}$ and $\widetilde{\Phi}^{>0}((\mathcal{C}_{31}^{\otimes 4})^{\omega_{(2)}}) \cap A(\omega_{(2)}) \neq \emptyset$. We put $B(\omega_{(2)}) := (\widetilde{\Phi}^{>0}((\mathcal{C}_{31}^{\otimes 4})^{\omega_{(2)}}) \setminus A(\omega_{(2)})) \cup \{X_j \mid 154 \leq j \leq 215\}$, where a direct computation shows that $\widetilde{\Phi}^{>0}((\mathcal{C}_{31}^{\otimes 4})^{\omega_{(2)}}) \setminus A(\omega_{(2)}) = \{X_j \mid 69 \leq j \leq 153\}$. Here, the monomials X_j , for all j , $1 \leq j \leq 215$, are determined as in the Appendix (see Subsect.3.4).

Proposition 2.2.4. $(Q_{31}^{\otimes 5})^{\omega_{(2)}}_{>0}$ is an $\mathbb{Z}_2$ -vector space of dimension 215 with the basis

$$S = [A(\omega_{(2)}) \cup B(\omega_{(2)})]_{\omega_{(2)}}.$$

Proof. Let $\omega := (2, 2, 2)$ be the weight vector of degree 14 and let X be an admissible monomial in $(\mathcal{C}_{31}^{\otimes 5})^{\omega_{(2)}}_{>0}$. Then $X = x_i x_j x_k Y^2$ with $1 \leq i < j < k \leq 5$, and $Y \in \mathbb{P}_{14}^{\otimes 5}$. Since $X \in (\mathcal{C}_{31}^{\otimes 5})^{\omega_{(2)}}$, due to Theorem 2.3, $Y \in (\mathcal{C}_{14}^{\otimes 5})^\omega$. By a direct computation, we see that if $Z \in (\mathcal{C}_{14}^{\otimes 5})^\omega$ and the monomial $T := x_i x_j x_k Z^2 \neq X_m$, for all m , $1 \leq m \leq 215$, then either T is one of the monomials as given in Lemmas 2.2.2, 2.2.3, or T is of the form $Z_1 Y_1^4$, where Z_1 is one of the following inadmissible monomials: $x_1 x_2^2 x_3^2 x_4 x_5$, $x_1^3 x_2^2 x_3 x_5$, $x_1^3 x_2^2 x_3 x_4$, $x_1^3 x_2^2 x_4 x_5$, $x_1^3 x_2^2 x_4 x_5$, $x_2^2 x_3^2 x_4 x_5$, $x_2^2 x_3 x_4 x_5^3$, $x_2^2 x_3 x_4^3 x_5$, $x_2^2 x_3^3 x_4 x_5$, $x_1^2 x_3 x_4 x_5^3$, $x_1^2 x_3 x_4^3 x_5$, $x_1^2 x_3^3 x_4 x_5$, $x_1^2 x_2 x_4 x_5^3$, $x_1^2 x_2 x_4^3 x_5$, $x_1^2 x_2 x_3 x_5^3$, $x_1^2 x_2 x_3 x_4^3$, $x_1^2 x_2 x_3^3 x_5$, $x_1^2 x_2 x_3^3 x_4$, $x_1^2 x_2^3 x_4 x_5$, $x_1^2 x_2^3 x_3 x_5$, $x_1^2 x_2^3 x_3 x_4$, $x_1^2 x_2 x_3 x_4 x_5^2$, $x_1^2 x_2 x_3 x_4^2 x_5$, $x_1^2 x_2 x_3^2 x_4 x_5$. Then, by Theorem 2.3, T is inadmissible. Since $X = x_i x_j x_k Y^2$ is admissible, we have $X = X_m$ for some m , $1 \leq m \leq 215$. Thus, the statement will be confirmed if the set S is linearly independent in $(Q_{31}^{\otimes 5})^{\omega_{(2)}}$. Indeed, assume that there is a linear relation $\mathcal{M} := \sum_{1 \leq m \leq 215} \gamma_m X_m \equiv_{\omega_{(2)}} 0$, whenever $\gamma_m \in \mathbb{Z}_2$. For each $(l, \mathcal{L}) \in \mathcal{N}_5$, we explicitly determine $\mathbf{p}_{(l, \mathcal{L})}(\mathcal{M})$ in terms of admissible monomials modulo $(\overline{\mathcal{A}}_2(\mathbb{P}_{31}^{\otimes 4})^{>0})$. Then, from the relations $\mathbf{p}_{(l, \mathcal{L})}(\mathcal{M}) \equiv_{\omega_{(2)}} 0$, for all $(l, \mathcal{L}) \in \mathcal{N}_5$, $\ell(\mathcal{L}) > 0$, and some other simple computations, we find that $\gamma_m = 0$, $\forall m$, $1 \leq m \leq 215$. The proposition is proved. $\square$

Remark 2.2.5. According to Sum [34], we have a direct summand decomposition of the $\mathbb{Z}_2$ -vector spaces: $Q_{31}^{\otimes 4} = (Q_{31}^{\otimes 4})^0 \oplus (Q_{31}^{\otimes 4})^{\omega_{(1)}}_{>0} \oplus (Q_{31}^{\otimes 4})^{\omega_{(2)}}_{>0}$. Since $(Q_{31}^{\otimes 4})^{\omega_{(1)}}_{>0} = \langle [x_1 x_2^2 x_3^4 x_4^{24}] \rangle$ and the powers of $x_1 x_2^2 x_3^4 x_4^{24}$ do not satisfy (1.4) in Conjecture 1.11, $\widetilde{\Phi}^{>0}((\mathcal{C}_{31}^{\otimes 4})^{\omega_{(1)}}) = \emptyset$. By a simple computation, one gets $\widetilde{\Phi}^{>0}((\mathcal{C}_{31}^{\otimes 4})^{\omega_{(2)}}) \subseteq (\mathcal{C}_{31}^{\otimes 5})^{\omega_{(2)}}_{>0}$, and

$$(\mathcal{C}_{31}^{\otimes 5})^{\omega_{(2)}}_{>0} = \widetilde{\Phi}^{>0}((\mathcal{C}_n^{\otimes 4})^{\omega_{(2)}}) \cup (A(\omega_{(2)}) \setminus \{X_j \mid 1 \leq j \leq 39\}) \cup \{X_j \mid 154 \leq j \leq 215\},$$

which implies that $\widetilde{\Phi}^{>0}((\mathcal{C}_n^{\otimes 4})^{\omega_{(2)}}) \subseteq (\mathcal{C}_{31}^{\otimes 5})^{\omega_{(2)}}_{>0} = A(\omega_{(2)}) \cup B(\omega_{(2)})$. Further, we see that if the weight vector $\omega \neq \omega_{(j)}$, $j = 1, 2$, then $(\mathcal{C}_{31}^{\otimes 4})^\omega = \emptyset$. So, by the proof of Proposition 2.2.6 below, it follows that $\widetilde{\Phi}_*((\mathcal{C}_{31}^{\otimes 4})^{\omega_{(3)}}) = \emptyset$. Combining these data, we may conclude that Conjecture 1.11 holds for $t = 5$ and degree 31.

Proposition 2.2.6. $(Q_{31}^{\otimes 5})^{\omega_{(3)}}_{>0}$ is an $\mathbb{Z}_2$ -vector space of dimension 70 with the basis

$$\mathcal{S} = [\{Y_j \mid 1 \leq j \leq 50\} \cup A(\omega_{(3)})]_{\omega_{(3)}},$$

where $A(\omega_{(3)}) = \{Y_j \mid 51 \leq j \leq 70\}$ and the monomials Y_j , $1 \leq j \leq 70$, are given as in the Subsect.3.5 of the Appendix.

Proof. Firstly, let i, j, k, l, m be five distinct integers such that $1 \leq i, j, k, l, m \leq 5$. Then, it is easy to see that the following monomials are inadmissible:

- (i) $T_1 := x_j^2 x_k^7 x_l^7 x_m^7$, $T_2 := x_j^3 x_k^6 x_l^7 x_m^7$, $T_3 := x_i x_j^2 x_k^6 x_l^7 x_m^7$,
- (ii) $T_4 := x_i^2 x_j^2 x_k^5 x_l^7 x_m^7$, $T_5 := x_i^2 x_j^3 x_k^4 x_l^7 x_m^7$, $T_6 := x_i^3 x_j^3 x_k^4 x_l^6 x_m^7$,
- (iii) $T_7 := x_i^7 x_j x_k^6 x_l^3 x_m^6$, $T_8 := x_i^7 x_j^3 x_k^6 x_l x_m^6$, $T_9 := x_i^7 x_j^6 x_k x_l^3 x_m^6$, $T_{10} := x_i^7 x_j^6 x_k^3 x_l x_m^6$, with $j < k < l$,
- (iv) $T_{11} := x_i^3 x_j^2 x_k^5 x_l^6 x_m^7$, $T_{12} := x_i^3 x_j^6 x_k^5 x_l^3 x_m^6$ with $j < k$,
- (v) $T_{13} := x_1^3 x_2^6 x_3^5 x_4^2 x_5^7$, $T_{14} := x_1^3 x_2^6 x_3^5 x_4^7 x_5^2$, $T_{15} := x_1^3 x_2^6 x_3^7 x_4^5 x_5^2$,
 $T_{16} := x_1^3 x_2^7 x_3^6 x_4^5 x_5^2$, $T_{17} := x_1^7 x_2^6 x_3^3 x_4^5 x_5^2$, $T_{18} := x_1^7 x_2^6 x_3^3 x_4^5 x_5^2$.

Consider the weight vector $\bar{\omega} := (4, 3, 1)$ of degree 14. Suppose that Y is an admissible monomial in $(\mathcal{C}_{31}^{\otimes 5})^{\omega_{(3)}^{>0}}$. Then $Y = x_a x_b x_c Z^2$ with $1 \leq a < b < c \leq 5$, and $Z \in \mathbb{P}_{14}^{\otimes 5}$. Since Y is admissible, according to Theorem 2.3, Z is admissible and $\omega(Z) = \bar{\omega}$. A direct computation shows that if Z is an admissible monomial in $\mathbb{P}_{14}^{\otimes 5}$ such that the monomial $x_a x_b x_c Z_1^2 \in (\mathbb{P}_{31}^{\otimes 5})^{\leq \omega_{(3)}}$ and $x_a x_b x_c Z_1^2 \neq Y_j$, for all j , $1 \leq j \leq 70$, then either $x_a x_b x_c Z_1^2$ of the form $T_d W^{2s}$, where s is a suitable integer and T_d is one of the above inadmissible monomials, $1 \leq d \leq 18$, or $x_a x_b x_c Z_1^2$ is one of the following monomials:

$$\begin{array}{cccc} x_1 x_2^3 x_3^6 x_4^{14} x_5^7, & x_1 x_2^3 x_3^{14} x_4^6 x_5^7, & x_1 x_2^3 x_3^{14} x_4^7 x_5^6, & x_1^3 x_2 x_3^6 x_4^{14} x_5^7, \\ x_1^3 x_2 x_3^{14} x_4^6 x_5^7, & x_1^3 x_2 x_3^{14} x_4^7 x_5^6, & x_1^3 x_2^5 x_3^2 x_4^{14} x_5^7, & x_1^3 x_2^5 x_3^{14} x_4^2 x_5^7, \\ x_1^3 x_2^5 x_3^{14} x_4^7 x_5^2, & x_1^3 x_2^{13} x_3^2 x_4^6 x_5^7, & x_1^3 x_2^{13} x_3^2 x_4^7 x_5^6, & x_1^3 x_2^{13} x_3^6 x_4^2 x_5^7, \\ x_1^3 x_2^{13} x_3^6 x_4^7 x_5^2, & x_1^3 x_2^{13} x_3^7 x_4^2 x_5^6, & x_1^3 x_2^{13} x_3^7 x_4^6 x_5^2, & x_1^3 x_2^5 x_3^{14} x_4^3 x_5^6, \\ x_1^3 x_2^5 x_3^{14} x_4^6 x_5^3, & x_1^3 x_2^{13} x_3^6 x_4^3 x_5^6, & x_1^3 x_2^{13} x_3^6 x_4^6 x_5^3, & x_1^3 x_2^5 x_3^6 x_4^{11} x_5^6, \\ x_1^3 x_2^5 x_3^6 x_4^{10} x_5^7, & x_1^3 x_2^5 x_3^{10} x_4^6 x_5^7, & x_1^3 x_2^5 x_3^{10} x_4^7 x_5^6. & \end{array}$$

These monomials are strictly inadmissible. Indeed, we will prove this statement for the monomials $x_1 x_2^3 x_3^6 x_4^{14} x_5^7$, and $x_1^3 x_2^{13} x_3^6 x_4^3 x_5^6$. The others can be computed by a similar technique. It is easily seen that $\omega(x_1 x_2^3 x_3^6 x_4^{14} x_5^7) = \omega(x_1^3 x_2^{13} x_3^6 x_4^3 x_5^6) = \omega_{(3)}$. Basing Cartan's formula, it is not too difficult to indicate that

$$\begin{aligned} x_1 x_2^3 x_3^6 x_4^{14} x_5^7 &= Sq^1 \left(x_1^2 x_2^3 x_3^5 x_4^7 x_5^{13} + x_1^2 x_2^3 x_3^5 x_4^{13} x_5^7 + x_1^2 x_2^5 x_3^5 x_4^7 x_5^{11} + x_1^2 x_2^5 x_3^5 x_4^{11} x_5^7 \right. \\ &\quad \left. + x_1^2 x_2^5 x_3^9 x_4^7 x_5^7 + x_1^2 x_2^9 x_3^5 x_4^7 x_5^7 + x_1^5 x_2 x_3^7 x_4^8 x_5^9 \right) \\ &\quad + Sq^2 \left(x_1 x_2^3 x_3^3 x_4^9 x_5^{13} + x_1 x_2^3 x_3^3 x_4^{13} x_5^9 + x_1 x_2^3 x_3^5 x_4^7 x_5^{13} + x_1 x_2^3 x_3^5 x_4^9 x_5^{11} \right. \\ &\quad \left. + x_1 x_2^3 x_3^5 x_4^{11} x_5^9 + x_1 x_2^3 x_3^5 x_4^{13} x_5^7 + x_1 x_2^5 x_3^5 x_4^7 x_5^{11} + x_1 x_2^5 x_3^5 x_4^{11} x_5^7 \right. \\ &\quad \left. + x_1 x_2^5 x_3^9 x_4^7 x_5^7 + x_1 x_2^9 x_3^5 x_4^7 x_5^7 \right) \\ &\quad + Sq^4 \left(x_1 x_2^3 x_3^3 x_4^9 x_5^{11} + x_1 x_2^3 x_3^3 x_4^{11} x_5^9 + x_1 x_2^5 x_3^5 x_4^7 x_5^9 + x_1 x_2^5 x_3^5 x_4^9 x_5^7 + x_1 x_2^6 x_3^6 x_4^7 x_5^7 \right) \\ &\quad + x_1 x_2^3 x_3^6 x_4^7 x_5^{14} \mod (\mathbb{P}_{31}^{\otimes 5})^{< \omega_{(3)}}, \\ x_1^3 x_2^{13} x_3^6 x_4^3 x_5^6 &= Sq^1 \left(x_1^3 x_2^7 x_3^5 x_4^5 x_5^{10} + x_1^3 x_2^7 x_3^5 x_4^6 x_5^9 + x_1^3 x_2^7 x_3^6 x_4^5 x_5^9 + x_1^3 x_2^7 x_3^6 x_4^9 x_5^5 \right. \\ &\quad \left. + x_1^3 x_2^7 x_3^9 x_4^6 x_5^5 + x_1^3 x_2^{11} x_3^5 x_4^6 x_5^5 + x_1^3 x_2^{14} x_3^3 x_4^5 x_5^5 + x_1^6 x_2^{11} x_3^3 x_4^5 x_5^5 \right) \\ &\quad + Sq^2 \left(x_1^3 x_2^{11} x_3^5 x_4^5 x_5^5 + x_1^3 x_2^{11} x_3^6 x_4^3 x_5^6 + x_1^3 x_2^{13} x_3^5 x_4^3 x_5^5 + x_1^5 x_2^7 x_3^3 x_4^5 x_5^9 \right. \\ &\quad \left. + x_1^5 x_2^7 x_3^3 x_4^9 x_5^5 + x_1^5 x_2^7 x_3^5 x_4^3 x_5^9 + x_1^5 x_2^9 x_3^3 x_4^3 x_5^9 + x_1^5 x_2^{11} x_3^3 x_4^5 x_5^5 \right) \\ &\quad + Sq^4 \left(x_1^3 x_2^7 x_3^3 x_4^5 x_5^9 + x_1^3 x_2^7 x_3^3 x_4^9 x_5^5 + x_1^3 x_2^7 x_3^5 x_4^3 x_5^9 + x_1^3 x_2^7 x_3^9 x_4^3 x_5^5 + x_1^3 x_2^9 x_3^3 x_4^3 x_5^9 \right. \\ &\quad \left. + x_1^3 x_2^9 x_3^5 x_4^5 x_5^5 + x_1^3 x_2^{11} x_3^3 x_4^5 x_5^5 + x_1^3 x_2^{11} x_3^5 x_4^3 x_5^5 \right) + x_1^3 x_2^{13} x_3^3 x_4^6 x_5^6 \mod (\mathbb{P}_{31}^{\otimes 5})^{< \omega_{(3)}}. \end{aligned}$$

The above computations, together with Theorem 2.3, allow us to claim that $x_a x_b x_c Z_1^2$ is inadmissible. Since $Y = x_a x_b x_c Z^2$ is admissible, we have $Y = Y_j$ for some j , $1 \leq j \leq 70$. We now prove the set $\mathcal{S}$ is linearly independent in $(Q_{31}^{\otimes 5})^{\omega_{(3)}}$. Suppose there is a linear relation $\mathcal{M} := \sum_{1 \leq j \leq 70} \gamma_j Y_j \equiv 0$, in which $\gamma_j \in \mathbb{Z}_2$ for $1 \leq j \leq 70$. By direct calculations using the relations $\mathbf{p}_{(\ell, \mathcal{L})}(\mathcal{M}) \equiv 0$, with $\ell(\mathcal{L}) \leq 2$, it may be concluded that $\gamma_j = 0$, for all j . The proof is completed. $\square$

2.3. Proof of Theorem 1.9

The proof of the theorem is proceeded through the following three steps.

Step 1. Firstly, a direct computation shows that if $X \in \mathcal{C}_{32}^{\otimes 5}$, then $\omega(X)$ is one of the sequences $\omega'_{(j)}$, for $1 \leq j \leq 7$, where $\omega'_{(4)} := (2, 1, 1, 3)$, $\omega'_{(5)} := (2, 3, 2, 2)$, $\omega'_{(6)} := (2, 3, 4, 1)$, and $\omega'_{(7)} :=$

$(4, 2, 4, 1)$. Indeed, it is easy to see that $x_1^{31}x_2$ is the minimal spike in $\mathbb{P}_{32}^{\otimes 5}$ and $\omega(x_1^{31}x_2) = \omega'_{(1)}$. Since $X \in \mathcal{C}_{32}^{\otimes 5}$ and $\deg(X) = 32$ even, by Theorem 2.5, either $\omega_1(X) = 2$ or $\omega_1(X) = 4$. If $\omega_1(X) = 2$, then $X = x_i x_j Y^2$ with $1 \leq i < j \leq 5$ and $Y \in \mathbb{P}_{15}^{\otimes 5}$. Since $X \in \mathcal{C}_{32}^{\otimes 5}$, by Theorem 2.3, $Y \in \mathcal{C}_{15}^{\otimes 5}$. According to Sum [33], we have $\omega(Y) \in \{(1, 1, 1, 1), (1, 1, 3), (3, 2, 2), (3, 4, 1)\}$ and so $\omega(X) \in \{\omega'_{(j)}\}_{j=1,4,5,6}$. In the case in which $\omega_1(X) = 4$, we have seen that by Theorem 2.3, X has the form $x_i x_j x_k x_\ell T^2$ with $T \in \mathcal{C}_{14}^{\otimes 5}$ and $1 \leq i < j < k < \ell \leq 5$. From the calculations of Ly-Tin [14], we get $\omega(T) \in \{(2, 2, 2), (2, 4, 1), (4, 3, 1)\}$ and therefore $\omega(X) \in \{\omega'_{(j)}\}_{j=2,3,7}$.

Step 2. Next, we will show that if $\omega(X) \in \{\omega'_{(j)}\}_{4 \leq j \leq 7}$, then the space $(Q_{32}^{\otimes 5})^{\omega(X)}$ is trivial. Indeed, from the works of Peterson [20], Kameko [13] and Sum [34], we deduce that

$$\dim(Q_{32}^{\otimes s})^{>0} = \begin{cases} 0, & \text{if } s = 1, \\ 3, & \text{if } s = 2, \\ 5, & \text{if } s = 3, \\ 57, & \text{if } s = 4. \end{cases}$$

The facts above show that

$$\dim(Q_{32}^{\otimes 5})^0 = \sum_{1 \leq s \leq 4} \binom{5}{s} \dim(Q_{32}^{\otimes s})^{>0} = 3 \times \binom{5}{2} + 5 \times \binom{5}{3} + 57 \times \binom{5}{4} = 365.$$

Furthermore, by a simple computation, we get $(\mathcal{C}_{32}^{\otimes 5})^0 = \bigcup_{1 \leq l \leq 5} \mathfrak{q}_{(l,5)}(\mathcal{C}_{32}^{\otimes 4}) = \bigcup_{1 \leq j \leq 3} (\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(j)})^0}$, where $|(\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(1)})^0}| = 115$, $|(\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(2)})^0}| = 175$ and $|(\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(3)})^0}| = 75$. Thus, it may be concluded that

$$\dim(Q_{32}^{\otimes 5})^{(\omega'_{(j)})^0} = \begin{cases} 115 & \text{if } j = 1, \\ 175 & \text{if } j = 2, \\ 75 & \text{if } j = 3, \\ 0 & \text{if } j = 4, 5, 6. \end{cases}$$

This means that we need only to prove $(Q_{32}^{\otimes 5})^{(\omega'_{(j)})^0} = 0$ for $j \in \{4, 5, 6\}$. Recall that if $\omega(X)$ is one of the sequences $\omega'_{(j)}$ for $j = 4, 5, 6$, then $X = x_i x_j Y^2$ with $1 \leq i < j \leq 5$ and $Y \in \mathcal{C}_{15}^{\otimes 5}$. When $\omega(X) = \omega'_{(4)}$, we see that X is one of the following monomials:

$$x_1^3 x_2^4 x_3^8 x_4^8 x_5^9, \quad x_1^3 x_2^4 x_3^8 x_4^9 x_5^8, \quad x_1^3 x_2^4 x_3^9 x_4^8 x_5^8, \quad x_1^3 x_2^5 x_3^8 x_4^8 x_5^8.$$

Due to Cartan's formula, one can easily see that

$$\begin{aligned} x_1^3 x_2^4 x_3^9 x_4^8 x_5^8 &= Sq^1(x_1^3 x_2 x_3^3 x_4^8 x_5^{16}) + Sq^2(x_1^5 x_2^2 x_3^3 x_4^4 x_5^{16}) + Sq^4(x_1^3 x_2^2 x_3^3 x_4^4 x_5^{16} + x_1^3 x_2^8 x_3^5 x_4^4 x_5^8) \\ &\quad + Sq^8(x_1^3 x_2^4 x_3^5 x_4^4 x_5^8) \pmod{(\mathbb{P}_{32}^{\otimes 5})^{<\omega'_{(4)}}} \end{aligned}$$

which implies that $x_1^3 x_2^4 x_3^9 x_4^8 x_5^8$ is $\omega'_{(4)}$ -decomposable. Similarly, the remaining monomials are also $\omega'_{(4)}$ -decomposable.

Now if $\omega(X) = \omega'_{(5)}$, then X is a permutation of the following monomials:

$$\begin{aligned} &x_1 x_2 x_3^2 x_4^{14} x_5^{14}, \quad x_1 x_2 x_3^6 x_4^{10} x_5^{14}, \quad x_1 x_2^2 x_3^2 x_4^{13} x_5^{14}, \quad x_1 x_2^2 x_3^3 x_4^{12} x_5^{14}, \\ &x_1 x_2^2 x_3^5 x_4^{10} x_5^{14}, \quad x_1 x_2^2 x_3^6 x_4^{10} x_5^{13}, \quad x_1 x_2^2 x_3^4 x_4^{10} x_5^{14}, \quad x_1 x_2^3 x_3^6 x_4^8 x_5^{14}, \\ &x_1 x_2^3 x_3^6 x_4^{10} x_5^{12}, \quad x_1^2 x_2^2 x_3^3 x_4^{12} x_5^{13}, \quad x_1^2 x_2^3 x_3^4 x_4^{10} x_5^{13}, \quad x_1^2 x_2^3 x_3^5 x_4^8 x_5^{14}, \\ &x_1^2 x_2^3 x_3^5 x_4^{10} x_5^{12}, \quad x_1^2 x_2^3 x_3^6 x_4^8 x_5^{13}, \quad x_1^3 x_2^4 x_3^5 x_4^{10} x_5^{10}, \quad x_1^3 x_2^5 x_3^6 x_4^8 x_5^{10}. \end{aligned}$$

These monomials are $\omega'_{(5)}$ -decomposable. Indeed, we prove this statement for the monomial $x_1^3 x_2^5 x_3^6 x_4^8 x_5^{10}$. The others can be computed by similar techniques. Applying Cartan's formula, we

get

$$\begin{aligned}
x_1^3 x_2^5 x_3^6 x_4^8 x_5^{10} &= Sq^1(x_1^3 x_2^3 x_3^9 x_4^4 x_5^{12} + x_1^3 x_2^3 x_3^{12} x_4^4 x_5^9 + x_1^6 x_2^3 x_3^9 x_4^4 x_5^9 + x_1^3 x_2^6 x_3^9 x_4^8 x_5^5) \\
&+ Sq^2(x_1^3 x_2^5 x_3^9 x_4^4 x_5^9 + x_1^3 x_2^3 x_3^{10} x_4^4 x_5^{10} + x_1^5 x_2^3 x_3^9 x_4^8 x_5^5) \\
&+ Sq^4(x_1^3 x_2^3 x_3^9 x_4^4 x_5^9 + x_1^3 x_2^3 x_3^9 x_4^8 x_5^5 + x_1^3 x_2^9 x_3^6 x_4^4 x_5^6) \\
&+ Sq^8(x_1^3 x_2^5 x_3^6 x_4^4 x_5^6) \mod (\mathbb{P}_{32}^{\otimes 5})^{<\omega'_{(5)}}.
\end{aligned}$$

Next if $\omega(X) = \omega'_{(6)}$ then a direct computation shows that X is of the form LY_1^{2r} , where r is a suitable integer and L is one of the following monomials:

$$\begin{aligned}
L_1 &= x_1^7 x_2^2 x_3^4 x_4^4 x_5^7, & L_2 &= x_1^7 x_2^2 x_3^4 x_4^7 x_5^4, & L_3 &= x_1^7 x_2^2 x_3^7 x_4^4 x_5^4, & L_4 &= x_1^7 x_2^7 x_3^2 x_4^4 x_5^4, \\
L_5 &= x_1^7 x_2^2 x_3^4 x_4^6 x_5^5, & L_6 &= x_1^7 x_2^2 x_3^6 x_4^4 x_5^5, & L_7 &= x_1^7 x_2^6 x_3^2 x_4^4 x_5^5, & L_8 &= x_1^7 x_2^2 x_3^4 x_4^5 x_5^6, \\
L_9 &= x_1^7 x_2^2 x_3^6 x_4^5 x_5^4, & L_{10} &= x_1^7 x_2^6 x_3^2 x_4^5 x_5^4, & L_{11} &= x_1^7 x_2^2 x_3^5 x_4^4 x_5^6, & L_{12} &= x_1^7 x_2^2 x_3^5 x_4^6 x_5^4, \\
L_{13} &= x_1^3 x_2^4 x_3^4 x_4^6 x_5^7, & L_{14} &= x_1^3 x_2^4 x_3^6 x_4^4 x_5^7, & L_{15} &= x_1^3 x_2^6 x_3^4 x_4^4 x_5^7, & L_{16} &= x_1^3 x_2^4 x_3^4 x_4^7 x_5^6, \\
L_{17} &= x_1^3 x_2^4 x_3^6 x_4^4 x_5^4, & L_{18} &= x_1^3 x_2^6 x_3^4 x_4^7 x_5^4, & L_{19} &= x_1^3 x_2^4 x_3^7 x_4^4 x_5^6, & L_{20} &= x_1^3 x_2^4 x_3^7 x_4^6 x_5^4, \\
L_{21} &= x_1^3 x_2^6 x_3^7 x_4^4 x_5^4, & L_{22} &= x_1^3 x_2^7 x_3^4 x_4^6 x_5^4, & L_{23} &= x_1^3 x_2^7 x_3^4 x_4^6 x_5^4, & L_{24} &= x_1^3 x_2^7 x_3^6 x_4^4 x_5^4, \\
L_{25} &= x_1^7 x_2^3 x_3^4 x_4^4 x_5^4, & L_{26} &= x_1^7 x_2^3 x_3^4 x_4^6 x_5^4, & L_{27} &= x_1^7 x_2^3 x_3^4 x_4^6 x_5^4, & L_{28} &= x_1^7 x_2^3 x_3^6 x_4^4 x_5^4, \\
L_{29} &= x_1^3 x_2^4 x_3^6 x_4^6 x_5^5, & L_{30} &= x_1^3 x_2^6 x_3^4 x_4^6 x_5^5, & L_{31} &= x_1^3 x_2^6 x_3^6 x_4^4 x_5^5, & L_{32} &= x_1^3 x_2^4 x_3^6 x_4^5 x_5^6, \\
L_{33} &= x_1^3 x_2^6 x_3^4 x_4^5 x_5^6, & L_{34} &= x_1^3 x_2^6 x_3^6 x_4^5 x_5^4, & L_{35} &= x_1^3 x_2^4 x_3^5 x_4^6 x_5^6, & L_{36} &= x_1^3 x_2^6 x_3^5 x_4^4 x_5^6, \\
L_{37} &= x_1^3 x_2^6 x_3^5 x_4^6 x_5^4, & L_{38} &= x_1^3 x_2^5 x_3^4 x_4^6 x_5^6, & L_{39} &= x_1^3 x_2^5 x_3^6 x_4^4 x_5^6, & L_{40} &= x_1^3 x_2^5 x_3^6 x_4^6 x_5^4.
\end{aligned}$$

It is not difficult to check that these monomials are inadmissible and so by Theorem 2.3, X is inadmissible.

Finally, if $\omega(X) = \omega'_{(7)}$ then $X = x_i x_j x_k x_\ell T^2$ with $1 \leq i < j < k < \ell \leq 5$ and $T \in \mathcal{C}_{14}^{\otimes 5}$. Then, by direct computations, we find that either $X = x_1^3 x_2^5 x_3^5 x_4^5 x_5^{14}$ or X is of the form LT_1^{2r} , where r is a suitable integer and L is one of the following monomials:

$$\begin{aligned}
L_{41} &= x_1^7 x_2^2 x_3^5 x_4^5 x_5^5, & L_{42} &= x_1^3 x_2^4 x_3^7 x_4^5 x_5^5, & L_{43} &= x_1^3 x_2^4 x_3^5 x_4^7 x_5^5, & L_{44} &= x_1^3 x_2^4 x_3^5 x_4^5 x_5^7, \\
L_{45} &= x_1^3 x_2^7 x_3^4 x_4^5 x_5^5, & L_{46} &= x_1^3 x_2^5 x_3^4 x_4^7 x_5^5, & L_{47} &= x_1^3 x_2^5 x_3^4 x_4^5 x_5^7, & L_{48} &= x_1^3 x_2^7 x_3^5 x_4^4 x_5^5, \\
L_{49} &= x_1^3 x_2^5 x_3^7 x_4^4 x_5^5, & L_{50} &= x_1^3 x_2^5 x_3^5 x_4^4 x_5^7, & L_{51} &= x_1^3 x_2^7 x_3^5 x_4^5 x_5^4, & L_{52} &= x_1^3 x_2^5 x_3^7 x_4^5 x_5^4, \\
L_{53} &= x_1^3 x_2^5 x_3^5 x_4^4 x_5^4, & L_{54} &= x_1^7 x_2^3 x_3^4 x_4^5 x_5^5, & L_{55} &= x_1^7 x_2^3 x_3^5 x_4^4 x_5^5, & L_{56} &= x_1^7 x_2^3 x_3^5 x_4^4 x_5^5, \\
L_{57} &= x_1^3 x_2^6 x_3^5 x_4^5 x_5^5, & L_{58} &= x_1^3 x_2^5 x_3^6 x_4^5 x_5^5, & L_{59} &= x_1^3 x_2^5 x_3^5 x_4^6 x_5^5.
\end{aligned}$$

Using the Cartan formula, it follows that

$$\begin{aligned}
x_1^3 x_2^5 x_3^5 x_4^5 x_5^{14} &= Sq^1(x_1^3 x_2^3 x_3^9 x_4^4 x_5^{13} + x_1^3 x_2^3 x_3^5 x_4^4 x_5^{17} + x_1^3 x_2^3 x_3^9 x_4^5 x_5^{11} + x_1^3 x_2^3 x_3^9 x_4^9 x_5^7) \\
&+ Sq^2(x_1^5 x_2^3 x_3^5 x_4^3 x_5^{14} + x_1^5 x_2^3 x_3^6 x_4^3 x_5^{13} + x_1^5 x_2^3 x_3^6 x_4^5 x_5^{11} \\
&\quad + x_1^5 x_2^3 x_3^9 x_4^4 x_5^7 + x_1^5 x_2^3 x_3^6 x_4^9 x_5^7) \\
&+ Sq^4(x_1^3 x_2^3 x_3^5 x_4^3 x_5^{14} + x_1^3 x_2^3 x_3^6 x_4^3 x_5^{13} + x_1^3 x_2^3 x_3^6 x_4^5 x_5^{11} \\
&\quad + x_1^3 x_2^3 x_3^9 x_4^6 x_5^7 + x_1^3 x_2^3 x_3^6 x_4^9 x_5^7) \mod (\mathbb{P}_{32}^{\otimes 5})^{<\omega'_{(7)}}
\end{aligned}$$

and so $X = x_1^3 x_2^5 x_3^5 x_4^5 x_5^{14}$ is $\omega'_{(7)}$ -decomposable. Further, it is easy to verify that L_i are inadmissible for $41 \leq i \leq 59$. Then, according to Theorem 2.3, X is inadmissible.

Now, we have seen that $\mathcal{C}_{32}^{\otimes 5} = \bigcup_{1 \leq j \leq 7} \mathcal{C}_{32}^{\otimes 5} \cap (\mathbb{P}_{32}^{\otimes 5})^{\leq \omega'_{(j)}} = \bigcup_{1 \leq j \leq 7} (\mathcal{C}_{32}^{\otimes 5})^{\omega'_{(j)}}$. On the other hand, we put $Q_{32}^{\otimes 5}(\omega'_{(j)}) := \langle [X] : X \in (\mathcal{C}_{32}^{\otimes 5})^{\omega'_{(j)}} \rangle \subset Q_{32}^{\otimes 5}$. Then, we have an isomorphism

$$Q_{32}^{\otimes 5}(\omega'_{(j)}) \rightarrow (Q_{32}^{\otimes 5})^{\omega'_{(j)}}, [X] \mapsto [X]_{\omega'_{(j)}}.$$

Combining this with the above computations gives

$$Q_{32}^{\otimes 5} = Q_{32}^{\otimes 5}(\omega'_{(1)}) \oplus Q_{32}^{\otimes 5}(\omega'_{(2)}) \oplus Q_{32}^{\otimes 5}(\omega'_{(3)}) \cong (Q_{32}^{\otimes 5})^{\omega'_{(1)}} \oplus (Q_{32}^{\otimes 5})^{\omega'_{(2)}} \oplus (Q_{32}^{\otimes 5})^{\omega'_{(3)}}.$$

Step 3. We now prove that $Q_{32}^{\otimes 5}$ is 1004-dimensional. It is suffice to determine that

$$\dim(Q_{32}^{\otimes 5})^{(\omega'_{(j)})^{>0}} = \begin{cases} 9, & \text{if } j = 1, \\ 310, & \text{if } j = 2, \\ 320, & \text{if } j = 3. \end{cases}$$

Indeed, for each $1 \leq l \leq 5$, let us consider the following sets:

$$\begin{aligned} B(31) &= \{x_l \mathbf{q}_{(l,5)}(X) : X \in (\mathcal{C}_{31}^{\otimes 4})^{>0}\}, \\ B(29) &= \{x_l^3 \mathbf{q}_{(l,5)}(X) : X \in (\mathcal{C}_{29}^{\otimes 4})^{>0}\}, \\ B(25) &= \{x_l^7 \mathbf{q}_{(l,5)}(X) : X \in (\mathcal{C}_{25}^{\otimes 4})^{>0}\}, \\ B(17) &= \{x_l^{15} \mathbf{q}_{(l,5)}(X) : X \in (\mathcal{C}_{17}^{\otimes 4})^{>0}\}, \\ B(1) &= \{x_l^{31} \mathbf{q}_{(l,5)}(X) : X \in (\mathcal{C}_1^{\otimes 4})^{>0}\}. \end{aligned}$$

For each $1 \leq j \leq 3$, writing $B(\omega'_{(j)}) = (\mathbb{P}^{\otimes 5})^{(\omega'_{(j)})^{>0}} \cap (B(31) \cup B(29) \cup B(25) \cup B(17) \cup B(1))$, in which $(\mathbb{P}^{\otimes 5})^{(\omega'_{(j)})^{>0}} := \langle \{X \in (\mathbb{P}^{\otimes 5})^{>0} : \omega(X) = \omega'_{(j)}\} \rangle$. By direct computations, one can easily obtain

$$\begin{aligned} B(\omega'_{(1)}) &= \{Z_j \mid 1 \leq j \leq 4\}, \\ B(\omega'_{(2)}) &= \{Z'_j \mid 1 \leq j \leq 196\}, \\ B(\omega'_{(3)}) &= \{Z''_j \mid 1 \leq j \leq 320\}. \end{aligned}$$

Notice that $\langle [B(\omega'_{(3)})]_{\omega'_{(3)}} \rangle$ is a $\mathbb{Z}_2$ -vector subspace of $(Q_{32}^{\otimes 5})^{(\omega'_{(3)})^{>0}}$. So, by Theorem 2.6, we conclude that $[B(\omega'_{(3)})]_{\omega'_{(3)}}$ is an basis of $(Q_{32}^{\otimes 5})^{(\omega'_{(3)})^{>0}}$. This means that $(Q_{32}^{\otimes 5})^{(\omega'_{(3)})^{>0}}$ is 320-dimensional. Now, we put $C(\omega'_{(1)}) = \{Z_j \mid 5 \leq j \leq 9\}$, and $C(\omega'_{(2)}) = \{Z'_j \mid 197 \leq j \leq 310\}$, where the monomials Z_j, Z'_j, Z''_j are described as in the Subjects 3.6, 3.7 and 3.8 of the Appendix. Then, the sets $[D(\omega'_{(1)}) := B(\omega'_{(1)}) \cup C(\omega'_{(1)})]_{\omega'_{(1)}}$ and $[D(\omega'_{(2)}) := B(\omega'_{(2)}) \cup C(\omega'_{(2)})]_{\omega'_{(2)}}$ are the bases of $(Q_{32}^{\otimes 5})^{(\omega'_{(1)})^{>0}}$ and $(Q_{32}^{\otimes 5})^{(\omega'_{(2)})^{>0}}$ respectively. Indeed, let $\omega := (1, 1, 1, 1)$ be the weight vector of degree 15 and let X be an admissible monomial in $(\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(1)})^{>0}}$. Then by Theorem 2.3, X has the form $x_i x_j Y^2$ with $1 \leq i < j \leq 5$, and $Y \in (\mathcal{C}_{15}^{\otimes 5})^\omega$. By direct calculations, we find that if $Y_1 \in (\mathcal{C}_{15}^{\otimes 5})^\omega$ and the monomial $T := x_i x_j Y_1^2 \neq Z_j$, for all j , $1 \leq j \leq 310$, then T is of the form LY_2^{16} , where L is one of the following inadmissible monomials: $x_1^3 x_2^4 x_3 x_4^8$, $x_1^3 x_2^4 x_3^8 x_4$, $x_1^3 x_2^4 x_3^8 x_5$. By Theorem 2.3, T is inadmissible. Since $X = x_i x_j Y^2$ is admissible, we have $X = Z_j$ for some j , $1 \leq j \leq 9$. Now, suppose that there is a linear relation $\mathcal{S} = \sum_{1 \leq j \leq 9} \gamma_j Z_j \equiv_{\omega'_{(1)}} 0$ where $\gamma_j \in \mathbb{Z}_2$ for every j . A simple computation using the relations $\mathbf{p}_{(1,k)}(\mathcal{S}) \equiv_{\omega'_{(1)}} 0$, $2 \leq k \leq 5$, one gets $\gamma_j = 0$, $j = 1, 2, \dots, 9$, and therefore $(Q_{32}^{\otimes 5})^{(\omega'_{(1)})^{>0}}$ is 9-dimensional.

Finally, to prove that the set $[D(\omega'_{(2)})]_{\omega'_{(2)}}$ is a basis of $(Q_{32}^{\otimes 5})^{(\omega'_{(2)})^{>0}}$, we first need the following.

Remark 2.3.1. The following monomials are strictly inadmissible:

$$\begin{array}{cccc} x_1^3 x_2^{12} x_3 x_4 x_5^{15}, & x_1^3 x_2^{12} x_3 x_4^{15} x_5, & x_1^3 x_2^{12} x_3^{15} x_4 x_5, & x_1^3 x_2^{15} x_3^{12} x_4 x_5, \\ x_1^{15} x_2^3 x_3^{12} x_4 x_5, & x_1^7 x_2^{11} x_3^{12} x_4 x_5, & x_1^3 x_2^{12} x_3 x_4^3 x_5^{13}, & x_1^3 x_2^{12} x_3^3 x_4 x_5^{13}, \\ x_1^3 x_2^{12} x_3^3 x_4^{13} x_5, & x_1^3 x_2^4 x_3 x_4^9 x_5^{15}, & x_1^3 x_2^4 x_3 x_4^{15} x_5^9, & x_1^3 x_2^4 x_3^9 x_4 x_5^{15}, \\ x_1^3 x_2^4 x_3^9 x_4^{15} x_5, & x_1^3 x_2^4 x_3^{15} x_4 x_5^9, & x_1^3 x_2^4 x_3^{15} x_4^9 x_5, & x_1^3 x_2^{15} x_3^4 x_4 x_5^9, \\ x_1^3 x_2^{15} x_3^4 x_4^9 x_5, & x_1^{15} x_2^3 x_3^4 x_4 x_5^9, & x_1^{15} x_2^3 x_3^4 x_4^9 x_5, & x_1^3 x_2^5 x_3^{14} x_4 x_5^9, \\ x_1^3 x_2^5 x_3^{14} x_4^9 x_5, & x_1^7 x_2^{11} x_3^4 x_4 x_5^9, & x_1^7 x_2^{11} x_3^4 x_4^9 x_5, & x_1^3 x_2^5 x_3^8 x_4^7 x_5^9. \end{array}$$

For simplicity, we prove this statement for the monomials $x_1^3 x_2^5 x_3^{14} x_4 x_5^9$ and $x_1^3 x_2^5 x_3^8 x_4^7 x_5^9$. The others can be proved by similar techniques. Indeed, by a direct computation using Cartan's formula,

we get

$$\begin{aligned}
x_1^3 x_2^5 x_3^{14} x_4 x_5^9 &= Sq^1 \left(x_1^3 x_2^3 x_3^7 x_4 x_5^{17} + x_1^3 x_2^3 x_3^{11} x_4 x_5^{13} + x_1^3 x_2^3 x_3^{13} x_4 x_5^{11} + x_1^3 x_2^3 x_3^{17} x_4 x_5^7 \right) \\
&\quad + Sq^2 \left(x_1^2 x_2^3 x_3^{11} x_4 x_5^{13} + x_1^2 x_2^3 x_3^{13} x_4 x_5^{11} + x_1^5 x_2^3 x_3^7 x_4 x_5^{14} + x_1^5 x_2^3 x_3^7 x_4 x_5^{13} \right. \\
&\quad \left. + x_1^5 x_2^3 x_3^7 x_4 x_5^{11} + x_1^5 x_2^3 x_3^7 x_4 x_5^8 + x_1^5 x_2^3 x_3^{11} x_4 x_5^7 + x_1^5 x_2^3 x_3^{13} x_4 x_5^7 \right. \\
&\quad \left. + x_1^5 x_2^3 x_3^{14} x_4 x_5^7 \right) \\
&\quad + Sq^4 \left(x_1^3 x_2^3 x_3^7 x_4 x_5^{14} + x_1^3 x_2^3 x_3^7 x_4 x_5^{13} + x_1^3 x_2^3 x_3^7 x_4 x_5^{11} + x_1^3 x_2^3 x_3^7 x_4 x_5^8 \right. \\
&\quad \left. + x_1^3 x_2^3 x_3^{11} x_4 x_5^7 + x_1^3 x_2^3 x_3^{13} x_4 x_5^7 + x_1^3 x_2^3 x_3^{14} x_4 x_5^7 \right) \\
&\quad + x_1^2 x_2^5 x_3^{11} x_4 x_5^{13} + x_1^2 x_2^5 x_3^{13} x_4 x_5^{11} + x_1^3 x_2^3 x_3^{12} x_4 x_5^{13} + x_1^3 x_2^3 x_3^{13} x_4 x_5^{12} \\
&\quad + x_1^3 x_2^4 x_3^{11} x_4 x_5^{13} + x_1^3 x_2^4 x_3^{13} x_4 x_5^{11} + x_1^3 x_2^5 x_3^7 x_4 x_5^9 + x_1^3 x_2^5 x_3^9 x_4 x_5^{14} \\
&\quad + x_1^3 x_2^5 x_3^9 x_4 x_5^{13} + x_1^3 x_2^5 x_3^9 x_4 x_5^{11} + x_1^3 x_2^5 x_3^9 x_4 x_5^8 + x_1^3 x_2^5 x_3^{11} x_4 x_5^9 \\
&\quad + x_1^3 x_2^5 x_3^{13} x_4 x_5^9 \mod (\mathbb{P}_{32}^{\otimes 5})^{<\omega'_{(2)}}, \\
x_1^3 x_2^5 x_3^8 x_4 x_5^9 &= Sq^1 \left(x_1^3 x_2^3 x_3^7 x_4 x_5^{17} + x_1^3 x_2^3 x_3^{11} x_4 x_5^{13} + x_1^3 x_2^3 x_3^{13} x_4 x_5^{11} + x_1^3 x_2^3 x_3^{17} x_4 x_5^7 \right) \\
&\quad + Sq^2 \left(x_1^2 x_2^3 x_3^{11} x_4 x_5^{13} + x_1^2 x_2^3 x_3^{13} x_4 x_5^{11} + x_1^5 x_2^3 x_3^7 x_4 x_5^{14} + x_1^5 x_2^3 x_3^7 x_4 x_5^{13} \right. \\
&\quad \left. + x_1^5 x_2^3 x_3^7 x_4 x_5^{11} + x_1^5 x_2^3 x_3^7 x_4 x_5^8 + x_1^5 x_2^3 x_3^{11} x_4 x_5^7 + x_1^5 x_2^3 x_3^{13} x_4 x_5^7 \right. \\
&\quad \left. + x_1^5 x_2^3 x_3^{14} x_4 x_5^7 \right) \\
&\quad + Sq^4 \left(x_1^3 x_2^3 x_3^7 x_4 x_5^{14} + x_1^3 x_2^3 x_3^7 x_4 x_5^{13} + x_1^3 x_2^3 x_3^7 x_4 x_5^{11} + x_1^3 x_2^3 x_3^7 x_4 x_5^8 \right. \\
&\quad \left. + x_1^3 x_2^3 x_3^{11} x_4 x_5^7 + x_1^3 x_2^3 x_3^{13} x_4 x_5^7 + x_1^3 x_2^3 x_3^{14} x_4 x_5^7 \right) \\
&\quad + x_1^2 x_2^5 x_3^{11} x_4 x_5^{13} + x_1^2 x_2^5 x_3^{13} x_4 x_5^{11} + x_1^3 x_2^3 x_3^{12} x_4 x_5^{13} + x_1^3 x_2^3 x_3^{13} x_4 x_5^{12} \\
&\quad + x_1^3 x_2^4 x_3^{11} x_4 x_5^{13} + x_1^3 x_2^4 x_3^{13} x_4 x_5^{11} + x_1^3 x_2^5 x_3^7 x_4 x_5^9 + x_1^3 x_2^5 x_3^9 x_4 x_5^{14} \\
&\quad + x_1^3 x_2^5 x_3^9 x_4 x_5^{13} + x_1^3 x_2^5 x_3^9 x_4 x_5^{11} + x_1^3 x_2^5 x_3^9 x_4 x_5^8 + x_1^3 x_2^5 x_3^{11} x_4 x_5^9 \\
&\quad + x_1^3 x_2^5 x_3^{13} x_4 x_5^9 \mod (\mathbb{P}_{32}^{\otimes 5})^{<\omega'_{(2)}}.
\end{aligned}$$

The above equalities show that $x_1^3 x_2^5 x_3^{14} x_4 x_5^9$ and $x_1^3 x_2^5 x_3^8 x_4 x_5^9$ are strictly inadmissible.

Next, suppose $X \in (\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(2)})^{>0}}$. Then, using Theorem 2.3, we have $X = x_a x_b x_c x_d L^2$ where $1 \leq a < b < c < d \leq 5$, and L is an admissible monomial in $(\mathbb{P}_{14}^{\otimes 5})^{\leq \omega}$ with $\omega = (2, 2, 2)$. A direct computation shows that if $L_1 \in (\mathcal{C}_{14}^{\otimes 5})^\omega$ and the monomial $M := x_a x_b x_c x_d L_1^2 \neq Z'_j$, for all j , $1 \leq j \leq 310$, then either M is one of the inadmissible monomials as shown in Remark 2.3.1 or M is of the form $x_1^3 x_2^2 x_3 x_4 x_5 L_2^4$, where $x_1^3 x_2^2 x_3 x_4 x_5$ is inadmissible. So, according to Theorem 2.3, M is inadmissible. Thus, there exists j for $1 \leq j \leq 310$ such that $X = Z'_j$. Now, we will prove that the set $[D(\omega'_{(2)})]_{\omega'_{(2)}}$ is linearly independent in $(Q_{32}^{\otimes 5})^{(\omega'_{(2)})^{>0}}$. Indeed, suppose there is a linear relation $\mathcal{S} = \sum_{1 \leq j \leq 310} \beta_j Z'_j \equiv_{\omega'_{(2)}} 0$ where $\beta_j \in \mathbb{Z}_2$ for all j . Acting the homomorphisms $\mathbf{p}_{(\ell, \mathcal{L})} : \mathbb{P}_{32}^{\otimes 5} \rightarrow \mathbb{P}_{32}^{\otimes 5}$ for $\ell(\mathcal{L}) \leq 2$ on the both sides of $\mathcal{S}$, we immediately obtain $\gamma_j = 0$ for any j , $1 \leq j \leq 310$. This completes the proof of the theorem.

Remark 2.3.2. By a simple computation, we have seen that $\widetilde{\Phi}^0((\mathcal{C}_{32}^{\otimes 4})^{\omega'_{(j)}}) \subset (\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(j)})^0}$ for $1 \leq j \leq 3$ and that $\widetilde{\Phi}^{>0}((\mathcal{C}_{32}^{\otimes 4})^{\omega'_{(j)}}) \subset B(\omega'_{(j)})$ for $j = 1, 3$ and $\widetilde{\Phi}^{>0}((\mathcal{C}_{32}^{\otimes 4})^{\omega'_{(2)}}) \cap B(\omega'_{(2)}) \neq \emptyset$.

Therefore, from the proof of Theorem 1.12, we have

$$\begin{aligned}\widetilde{\Phi}^{>0}((\mathcal{C}_{32}^{\otimes 4})^{\omega'_{(j)}}) &\subset (\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(j)})^{>0}} \text{ for } j = 1, 3, \\ (\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(2)})^{>0}} &= \widetilde{\Phi}^{>0}((\mathcal{C}_{32}^{\otimes 4})^{\omega'_{(2)}}) \cup B(\omega'_{(2)}) \cup \{Z'_j \mid 285 \leq j \leq 310\}.\end{aligned}$$

Thus, Conjecture 1.11 holds also in the 5-variable case and the degree 32.

2.4. Proof of Theorem 1.12

Obviously $n_d = 5(2^d - 1) + 21 \cdot 2^d$. In [26], we have shown that $Q_{n_d}^{\otimes 5}$ is 1894-dimensional for arbitrary $d > 0$. Let us now consider $\zeta = 1 < d$. It is easy to check that

$$\mu(n_\zeta) = 3 = \alpha(n_\zeta + \mu(n_\zeta)).$$

So, invoking Theorem 2.7, we may conclude that

$$\dim Q_{5(2^{d+4}-1)+n_1 \cdot 2^{d+4}}^{\otimes 6} = (2^6 - 1) \times \dim Q_{n_d}^{\otimes 5} = 63 \times 1894 = 119322,$$

for any positive integer d . The theorem is proved.

2.5. Proof of Theorem 1.13

Before coming to the proof of the theorem, we give the following definition: Let $\mathbb{Z}_2^5$ be a rank 5 elementary abelian 2-group, which views as a 5-dimensional $\mathbb{Z}_2$ -vector space. As well known, $H^*(\mathbb{Z}_2^5, \mathbb{Z}_2) \cong S((\mathbb{Z}_2^5)^*)$, the symmetric algebra over the dual space $(\mathbb{Z}_2^5)^* = H^1(\mathbb{Z}_2^5, \mathbb{Z}_2)$. Pick $x_1, \dots, x_5$ to be a basis of $H^1(\mathbb{Z}_2^5, \mathbb{Z}_2)$. Then, one has an isomorphism $H^*(\mathbb{Z}_2^5, \mathbb{Z}_2) \cong \mathbb{P}^{\otimes 5} = \mathbb{Z}_2[x_1, \dots, x_5]$. One should note that $\mathbb{Z}_2^5 \cong \langle x_1, \dots, x_5 \rangle \subset \mathbb{P}^{\otimes 5}$. For $1 \leq d \leq 5$, we define the $\mathbb{Z}_2$ -linear map $\sigma_d : \mathbb{Z}_2^5 \rightarrow \mathbb{Z}_2^5$ by setting $\sigma_d(x_d) = x_{d+1}$, $\sigma_d(x_{d+1}) = x_d$, $\sigma_d(x_i) = x_i$ for $i \neq d, d+1$, $1 \leq d \leq 4$, and $\sigma_5(x_1) = x_1 + x_2$, $\sigma_5(x_i) = x_i$ for $2 \leq i \leq 5$.

Remark 2.5.1. Denote by Σ_5 the symmetric group of degree 5. Then, Σ_5 is generated by the ones associated with $\sigma_1, \dots, \sigma_4$. For each permutation in Σ_5 , consider corresponding permutation matrix; these form a group of matrices isomorphic to Σ_5 . Indeed, consider the following map $\Delta : \Sigma_5 \rightarrow \mathcal{P}_{5 \times 5}$, where the latter is the set of permutation matrices of order 5. This map is defined as follows: given $\sigma \in \Sigma_5$, the i -th column of $\Delta(\sigma)$ is the column vector with a 1 in the $\rho(i)$ -th position, and 0 elsewhere. It is easy to see that $\Delta(\rho)$ is indeed a permutation matrix, since a 1 occurs in any position if and only if that position is described by $(\rho(i), i)$, for any $1 \leq i \leq 5$. The map Δ is clearly multiplicative. (It is to be noted that because these are matrices, it is enough to show that each corresponding entry is equal. So let us take the entry (i, j) of each matrix.) Then, $\Delta(\rho \circ \rho')_{ij} = 1$ if and only if $i = \rho \circ \rho'(j)$. Note also that by ordinary matrix multiplication, one has $(\Delta(\rho)\Delta(\rho'))_{ij} = \sum_{1 \leq k \leq 5} \Delta(\rho)_{ik} \Delta(\rho')_{kj}$. Now, we know that $\Delta(\rho)_{ik} = 1$ only when $i = \rho(k)$. Similarly, $\Delta(\rho')_{kj} = 1$ only when $k = \rho'(j)$. Hence, their product is one precisely when both of these happen: $i = \rho(k)$, and $k = \rho'(j)$. If both these do not happen simultaneously, then whenever one of $\Delta(\rho)_{ik}$, $\Delta(\rho')_{kj}$ is one of the other will be zero, so the whole sum will be zero. However, this is the same as saying that the sum is one exactly when $i = \rho \circ \rho'(j)$. This description matches with the description given for $\Delta(\rho \circ \rho')_{ij}$ given earlier. Hence, entry by entry these matrices are the same. Therefore the matrices are the same, and hence Δ is a homomorphism between the two spaces, an isomorphism as it has trivial kernel and the sets are of the same cardinality. Thus, $GL_5 \cong GL(\mathbb{Z}_2^5)$, and GL_5 is generated by the matrices associated with $\sigma_1, \dots, \sigma_5$. Let $X = x_1^{a_1} x_2^{a_2} \dots x_5^{a_5}$ be a monomial in $\mathbb{P}^{\otimes 5}$. Then, the weight vector $\omega(X)$ is invariant under the permutation of the generators x_j , $j = 1, 2, \dots, 5$; hence $(Q^{\otimes 5})^{\omega(X)}$ also has a Σ_5 -module structure. We see that the linear map σ_d induces a homomorphism of $\mathcal{A}_2$ -algebras which is also denoted by $\sigma_d : \mathbb{P}^{\otimes 5} \rightarrow \mathbb{P}^{\otimes 5}$. So, a class $[X]_\omega \in (Q^{\otimes 5})^\omega$ is an GL_5 -invariant if and only if $\sigma_d(X) \equiv_\omega X$ for $1 \leq d \leq 5$. If $\sigma_d(X) \equiv_\omega X$ for $1 \leq d \leq 4$, then $[X]_\omega$ is an Σ_5 -invariant.

We are now ready to prove Theorem 1.13. In what follows: Let ω be a weight vector of degree n and let $F_1, F_2, \dots, F_s$ be the monomials in $(\mathbb{P}_n^{\otimes 5})^\omega$. For a subgroup G of $GL_5(\mathbb{Z}_2)$, denote by $G(F_1; F_2; \dots, F_s)$ the G -submodule of $(Q_n^{\otimes 5})^\omega$ generated by the set $\{[F_i]_\omega : 1 \leq i \leq s\}$. We also note that if ω is a weight vector of a minimal spike, then $[F_i]_\omega = [F_i]$ for all i .

Proof of Case $n = 14$. Firstly, we have the following remarks.

Remark 2.5.2. According to Ly-Tin [14], if X is an admissible monomial in $\mathbb{P}_{14}^{\otimes 5}$, then $\omega(X)$ is one of the following sequences: $\tilde{\omega}_{(1)} := (2, 2, 2)$, $\tilde{\omega}_{(2)} := (2, 4, 1)$ and $\tilde{\omega}_{(3)} := (4, 3, 1)$. A direct computation using a result in Sum [34], we have

$$(\mathcal{C}_{14}^{\otimes 5})^0 = \bigcup_{1 \leq l \leq 5} \mathbf{q}_{(l,5)}(\mathcal{C}_{14}^{\otimes 4}) = (\mathcal{C}_{14}^{\otimes 5})^{\tilde{\omega}_{(1)}} \cup (\mathcal{C}_{14}^{\otimes 5})^{\tilde{\omega}_{(3)}},$$

where $|(\mathcal{C}_{14}^{\otimes 5})^{\tilde{\omega}_{(1)}}| = 115$ and $|(\mathcal{C}_{14}^{\otimes 5})^{\tilde{\omega}_{(3)}}| = 75$. For each $1 \leq l \leq 5$, letting

$$\begin{aligned} E(13) &= \{x_l \mathbf{q}_{(l,5)}(X) : X \in (\mathcal{C}_{13}^{\otimes 4})^{>0}\}, \\ E(11) &= \{x_l^3 \mathbf{q}_{(l,5)}(X) : X \in (\mathcal{C}_{11}^{\otimes 4})^{>0}\}, \\ E(7) &= \{x_l^7 \mathbf{q}_{(l,5)}(X) : X \in (\mathcal{C}_7^{\otimes 4})^{>0}\}, \end{aligned}$$

and denote $E(\tilde{\omega}_{(j)}) = (\mathbb{P}^{\otimes 5})^{\tilde{\omega}_{(j)}} \cap (E(13) \cup E(11) \cup E(7))$, with

$$(\mathbb{P}^{\otimes 5})^{\tilde{\omega}_{(j)}} := \langle \{X \in (\mathbb{P}_{14}^{\otimes 5})^{>0} : \omega(X) = \tilde{\omega}_{(j)}\} \rangle$$

for $j = 2, 3$. Then, by a simple computation, it follows that

$$\begin{aligned} (\mathcal{C}_{14}^{\otimes 5})^{\tilde{\omega}_{(1)}} &= \widetilde{\Phi}^{>0}((\mathcal{C}_{14}^{\otimes 4})^{\tilde{\omega}_{(1)}}) \cup \{x_1 x_2^2 x_3^4 x_4^3 x_5^4, x_1^3 x_2^4 x_3 x_4^2 x_5^4\} \\ (\mathcal{C}_{14}^{\otimes 5})^{\tilde{\omega}_{(2)}} &= E(\tilde{\omega}_{(2)}) \cup D, \\ (\mathcal{C}_{14}^{\otimes 5})^{\tilde{\omega}_{(3)}} &= E(\tilde{\omega}_{(3)}), \end{aligned}$$

where D is the set of the following admissible monomials:

$$\begin{array}{cccc} x_1 x_2^2 x_3^2 x_4^3 x_5^6, & x_1 x_2^2 x_3^3 x_4^2 x_5^6, & x_1 x_2^2 x_3^3 x_4^6 x_5^2, & x_1 x_2^3 x_3^2 x_4^2 x_5^6, \\ x_1 x_2^3 x_3^2 x_4^6 x_5^2, & x_1 x_2^3 x_3^6 x_4^2 x_5^2, & x_1^3 x_2 x_3^2 x_4^6 x_5^6, & x_1^3 x_2 x_3^2 x_4^6 x_5^2, \\ x_1^3 x_2 x_3^6 x_4^2 x_5^2, & x_1^3 x_2^5 x_3^2 x_4^2 x_5^2. & & \end{array}$$

So, we have

$$\begin{aligned} (\mathcal{C}_{14}^{\otimes 5})^{\tilde{\omega}_{(1)}} &= \{W_j | 1 \leq j \leq 130\}, \\ (\mathcal{C}_{14}^{\otimes 5})^{\tilde{\omega}_{(2)}} &= (\mathcal{C}_{14}^{\otimes 5})^{\tilde{\omega}_{(2)}}^{>0} = \{W_j | 131 \leq j \leq 145\}, \\ (\mathcal{C}_{14}^{\otimes 5})^{\tilde{\omega}_{(3)}} &= \{W_j | 146 \leq j \leq 320\}, \end{aligned}$$

where the monomials W_j for $1 \leq j \leq 320$, are determined as in Subsects.3.1, 3.2 and 3.3 of Appendix. Therefore, we have an isomorphism

$$Q_{14}^{\otimes 5} \cong (Q_{14}^{\otimes 5})^{\tilde{\omega}_{(1)}} \oplus (Q_{14}^{\otimes 5})^{\tilde{\omega}_{(2)}} \oplus (Q_{14}^{\otimes 5})^{\tilde{\omega}_{(3)}},$$

where

$$\dim(Q_{14}^{\otimes 5})^{\tilde{\omega}_{(j)}} = \begin{cases} 130 & \text{if } j = 1, \\ 15 & \text{if } j = 2, \\ 175 & \text{if } j = 3, \end{cases}$$

and so $Q_{14}^{\otimes 5}$ has dimension 320. This result has also been computed by Ly-Tin [14]. However, our approach above is much less computational. Further, we observe that $\tilde{\Phi}_*((\mathcal{C}_{14}^{\otimes 4})^{\tilde{\omega}_{(j)}}) \subset (\mathcal{C}_{14}^{\otimes 5})^{\tilde{\omega}_{(j)}}$ for $j = 1, 3$ and $\tilde{\Phi}_*((\mathcal{C}_{14}^{\otimes 4})^{\tilde{\omega}_{(2)}}) = \emptyset$ since $(\mathcal{C}_{14}^{\otimes 4})^{\tilde{\omega}_{(2)}} = \emptyset$ and so Conjecture 1.11 satisfies for $m = 5$ and the degree 14.

Now, to prove that $[P_{\mathcal{A}_2}((\mathbb{P}_{14}^{\otimes 5})^*)]_{GL_5}$ is 1-dimensional, we need determine the spaces of GL_5 -invariants $[(Q_{14}^{\otimes 5})^{\tilde{\omega}_{(j)}}]^{GL_5}$ for $1 \leq j \leq 3$. We obtain the following.

Proposition 2.5.3. *We have*

$$[(Q_{14}^{\otimes 5})^{\tilde{\omega}_{(j)}}]^{GL_5} = \begin{cases} \langle [\zeta] \rangle & \text{if } j = 1, \\ 0 & \text{if } j = 2, 3, \end{cases}$$

where

$$\begin{aligned} \zeta &= W_{51} + W_{53} + W_{55} + W_{56} + W_{57} + W_{111} + W_{112} + W_{113} + W_{114} + W_{115} \\ &\quad + W_{116} + W_{117} + W_{118} + W_{119} + W_{121} + W_{122} + W_{124} + W_{125} \\ &\quad + W_{127} + W_{128} + W_{129} + W_{130}. \end{aligned}$$

Consequently, $\dim[P_{\mathcal{A}_2}((\mathbb{P}_{14}^{\otimes 5})^*)]_{GL_5} \leq 1$.

Proof. We prove the proposition for the cases $j = 1, 2$. The result when $j = 3$ can be obtained by using similar idea. We first consider the case $j = 1$. Note that since $\tilde{\omega}_{(1)}$ is the weight vector of the minimal spike $x_1^7 x_2^7 \in \mathbb{P}_{14}^{\otimes 5}$, $[W_j]_{\tilde{\omega}_{(1)}} = [W_j]$ for all $W_j \in (Q_{14}^{\otimes 5})^{\tilde{\omega}_{(1)}}$. From the admissible monomial basis of $(Q_{14}^{\otimes 5})^{\tilde{\omega}_{(1)}}$, we have a direct summand decomposition of the Σ_5 -submodules:

$$\begin{aligned} (Q_{14}^{\otimes 5})^{\tilde{\omega}_{(1)}} &= \Sigma_5(W_1) \oplus \Sigma_5(W_{11}) \oplus \Sigma_5(W_{41}) \oplus \Sigma_5(W_{61}) \\ &\quad \oplus \Sigma_5(W_{51}, W_{81}) \oplus \mathbb{V}, \end{aligned}$$

where

$$\begin{aligned} \Sigma_5(W_1) &= \langle \{[W_j] : 1 \leq j \leq 10\} \rangle, \quad \Sigma_5(W_{11}) = \langle \{[W_j] : 11 \leq j \leq 40\} \rangle, \\ \Sigma_5(W_{41}) &= \langle \{[W_j] : 41 \leq j \leq 50\} \rangle, \quad \Sigma_5(W_{61}) = \langle \{[W_j] : 61 \leq j \leq 80\} \rangle, \\ \Sigma_5(W_{51}, W_{81}) &= \langle \{[W_j] : j = 51, 52, \dots, 60, 81, \dots, 115\} \rangle, \\ \mathbb{V} &= \langle \{[W_j] : 116 \leq j \leq 130\} \rangle. \end{aligned}$$

Then, a direct computation shows that

- (i) $[\Sigma_5(W_1)]^{\Sigma_5} = \langle [\widehat{p}_1] \rangle$ with $\widehat{p}_1 := \sum_{1 \leq j \leq 10} W_j$,
- (ii) $[\Sigma_5(W_{11})]^{\Sigma_5} = \langle [\widehat{p}_2] \rangle$ with $\widehat{p}_2 := \sum_{11 \leq j \leq 40} W_j$,
- (iii) $[\Sigma_5(W_{41})]^{\Sigma_5} = \langle [\widehat{p}_3] \rangle$ with $\widehat{p}_3 := \sum_{41 \leq j \leq 50} W_j$,
- (iv) $[\Sigma_5(W_{51}, W_{81})]^{\Sigma_5} = \langle [\widehat{p}_4] \rangle$
with $\widehat{p}_4 := W_{51} + W_{53} + W_{55} + W_{56} + W_{57} + \sum_{111 \leq j \leq 115} W_j$,
- (v) $[\Sigma_5(W_{61})]^{\Sigma_5} = \langle [\widehat{p}_5] \rangle$ with $\widehat{p}_5 := \sum_{61 \leq j \leq 80} W_j$,
- (vi) $[\mathbb{V}]^{\Sigma_5} = \langle [\widehat{p}_6] \rangle$ with $\widehat{p}_6 := \sum_{116 \leq j \leq 119} W_j + W_{121} + W_{122} + W_{124} + W_{125} + \sum_{127 \leq j \leq 130} W_j$.

Indeed, for simplicity, we prove the result (vi) in detail. The others can be proved by the similar computations. An admissible monomial basis for $\mathbb{V}$ is the set $\{[W_j] : 116 \leq j \leq 130\}$. Suppose that $[F] \in [\mathbb{V}]^{\Sigma_5}$, then

$$F \equiv \sum_{116 \leq j \leq 130} \gamma_j W_j$$

with $\gamma_j \in \mathbb{Z}_2$. Acting the homomorphisms $\sigma_d : \mathbb{P}^{\otimes 5} \rightarrow \mathbb{P}^{\otimes 5}$ on both sides of this equality and using the relations $\sigma_d(F) + F \equiv 0$, for $1 \leq d \leq 4$, we get the equalities below:

$$\begin{aligned}
\sigma_1(F) + F &\equiv (\gamma_{119} + \gamma_{121})W_{116} + \gamma_{120}W_{117} + \gamma_{123}W_{118} + (\gamma_{122} + \gamma_{130})(W_{122} + W_{130}) \\
&\quad + (\gamma_{125} + \gamma_{127})(W_{125} + W_{127}) + (\gamma_{126} + \gamma_{128} + \gamma_{130})W_{126} \\
&\quad + (\gamma_{122} + \gamma_{126} + \gamma_{128})W_{128} \equiv 0, \\
\sigma_2(F) + F &\equiv (\gamma_{116} + \gamma_{119})(W_{116} + W_{119}) + (\gamma_{117} + \gamma_{120} + \gamma_{129})(W_{117} + W_{120}) \\
&\quad + (\gamma_{118} + \gamma_{122})(W_{118} + W_{122}) + (\gamma_{123} + \gamma_{126})(W_{123} + W_{126}) \\
&\quad + (\gamma_{124} + \gamma_{125} + \gamma_{126})W_{124} + (\gamma_{123} + \gamma_{124} + \gamma_{125})W_{125} \\
&\quad + (\gamma_{128} + \gamma_{130})(W_{128} + W_{130}) \equiv 0, \\
\sigma_3(F) + F &\equiv (\gamma_{117} + \gamma_{118} + \gamma_{123})W_{117} + (\gamma_{117} + \gamma_{118} + \gamma_{120})W_{118} \\
&\quad + (\gamma_{119} + \gamma_{121})(W_{119} + W_{121}) + (\gamma_{120} + \gamma_{123})(W_{120} + W_{123}) \\
&\quad + (\gamma_{124} + \gamma_{129})(W_{124} + W_{129}) + (\gamma_{122} + \gamma_{125} + \gamma_{126})(W_{125} + W_{126}) \\
&\quad + (\gamma_{127} + \gamma_{128})(W_{127} + W_{128}) \equiv 0, \\
\sigma_4(F) + F &\equiv (\gamma_{116} + \gamma_{117})(W_{116} + W_{117}) + (\gamma_{119} + \gamma_{120} + \gamma_{129})W_{119} \\
&\quad + (\gamma_{119} + \gamma_{120} + \gamma_{121})W_{120} + (\gamma_{121} + \gamma_{129})(W_{121} + W_{129}) \\
&\quad + \gamma_{123}W_{124} + \gamma_{126}W_{125} + (\gamma_{128} + \gamma_{130})W_{127} \equiv 0.
\end{aligned}$$

Thus $\gamma_{120} = \gamma_{123} = \gamma_{126} = 0$ and $\gamma_j = \gamma_{121}$ for $j \neq 120, 123, 126$. This confirms the above statement. Now, let $G \in (\mathbb{P}_{14}^{\otimes 5})^{\widetilde{\omega}_{(1)}}$ such that $[G] \in [(Q_{14}^{\otimes 5})^{\widetilde{\omega}_{(1)}}]^{GL_5}$. Because $\Sigma_5 \subset GL_5$, from the above calculations, we have

$$G \equiv \beta_1 \widehat{p}_1 + \beta_2 \widehat{p}_2 + \beta_3 \widehat{p}_3 + \beta_4 \widehat{p}_4 + \beta_5 \widehat{p}_5 + \beta_6 \widehat{p}_6$$

in which $\beta_j \in \mathbb{Z}_2$. Using the relation $\sigma_5(G) + G \equiv 0$, we get

$$(\beta_1 + \beta_2)W_4 + (\beta_2 + \beta_5)W_{14} + \beta_2 W_{20} + (\beta_2 + \beta_3)W_{32} + (\beta_4 + \beta_6)W_{111} + \text{other terms} \equiv 0.$$

This equality implies that $\beta_4 = \beta_6$ and $\beta_j = 0$ for $j \neq 4, 6$. This means that

$$[(Q_{14}^{\otimes 5})^{\widetilde{\omega}_{(1)}}]^{GL_5} = \langle [\zeta := \widehat{p}_4 + \widehat{p}_6] \rangle.$$

Now we prove the proposition for the case $j = 2$. According to Remark 2.5.2, an admissible basis of $(Q_{14}^{\otimes 5})^{\widetilde{\omega}_{(2)}}$ is the set $\{[W_j]_{\widetilde{\omega}_{(2)}} : 131 \leq j \leq 145\}$. From this, a direct computation shows that

$$(Q_{14}^{\otimes 5})^{\widetilde{\omega}_{(2)}} = (Q_{14}^{\otimes 5})^{\widetilde{\omega}_{(2)} > 0} \cong \Sigma_5(W_{131}) \bigoplus \Sigma_5(W_{136}),$$

where $\Sigma_5(W_{131}) = \langle \{[W_j]_{\widetilde{\omega}_{(2)}} : 131 \leq j \leq 135\} \rangle$ and $\Sigma_5(W_{136}) = \langle \{[W_j]_{\widetilde{\omega}_{(2)}} : 136 \leq j \leq 145\} \rangle$ and that

$$\begin{aligned}
[\Sigma_5(W_{131})]^{\Sigma_5} &= \langle [\sum_{131 \leq j \leq 135} W_j]_{\widetilde{\omega}_{(2)}} \rangle \\
[\Sigma_5(W_{136})]^{\Sigma_5} &= \langle [\sum_{136 \leq j \leq 145} W_j]_{\widetilde{\omega}_{(2)}} \rangle.
\end{aligned}$$

Then if $H \in (\mathbb{P}_{14}^{\otimes 5})^{\widetilde{\omega}_{(2)}}$ such that $[H] \in [(Q_{14}^{\otimes 5})^{\widetilde{\omega}_{(2)}}]^{GL_5}$, then

$$H \equiv_{\widetilde{\omega}_{(2)}} \xi_1 \sum_{131 \leq j \leq 135} W_j + \xi_2 \sum_{136 \leq j \leq 145} W_j$$

with ξ_1 and ξ_2 belong to $\mathbb{Z}_2$. Straightforward calculations indicate that

$$\sigma_5(H) + H \equiv_{\widetilde{\omega}_{(2)}} (\beta_1 + \beta_2)W_{134} + \beta_2(W_{139} + W_{140} + W_{141}) \equiv_{\widetilde{\omega}_{(2)}} 0$$

which implies $\xi_1 = \xi_2 = 0$. So $[(Q_{14}^{\otimes 5})^{\widetilde{\omega}_{(2)}}]^{GL_5}$ is trivial.

Now because $Q_{14}^{\otimes 5} \cong \bigoplus_{1 \leq j \leq 3} (Q_{14}^{\otimes 5})^{\widetilde{\omega}_{(j)}}$ (see Remark 2.5.2) and $[P_{\mathcal{A}_2}((\mathbb{P}_{14}^{\otimes 5})^*)]_{GL_5} \cong [Q_{14}^{\otimes 5}]^{GL_5}$, by the above calculations, we conclude that

$$\dim[P_{\mathcal{A}_2}((\mathbb{P}_{14}^{\otimes 5})^*)]_{GL_5} \leq \sum_{1 \leq j \leq 3} \dim[(Q_{14}^{\otimes 5})^{\widetilde{\omega}_{(j)}}]^{GL_5} = 1.$$

The proof of the proposition is completed. $\square$

Now following Lin [15] and Chen [7], $\text{Ext}_{\mathcal{A}_2}^{5,5+14}(\mathbb{Z}_2, \mathbb{Z}_2) = \langle h_0 d_0 \rangle$ with $h_0 d_0 \neq 0$. Further, $h_0 \in \text{Im}(Tr_1^{\mathcal{A}_2})$ (see Singer [30]), $d_0 \in \text{Im}(Tr_4^{\mathcal{A}_2})$ (see Hà [9]) and the "total" transfer $Tr^{\mathcal{A}_2}$ is an algebraic homomorphism. So, $h_0 d_0 \in \text{Im}(Tr_5^{\mathcal{A}_2})$, that is, $\dim[P_{\mathcal{A}_2}((\mathbb{P}_{14}^{\otimes 5})^*)]_{GL_5} \geq 1$. This and Proposition 2.5.3 show that $[P_{\mathcal{A}_2}((\mathbb{P}_{14}^{\otimes 5})^*)]_{GL_5}$ is one-dimensional, which confirms the result for the case $n = 14$. $\square$

Proof of Case $n = 31$. Let us recall that Kameko's squaring operation $\widetilde{Sq}_*^0 : Q_{31}^{\otimes 5} \rightarrow Q_{13}^{\otimes 5}$ is an epimorphism of $\mathbb{Z}_2 GL_5$ -modules. So, $Q_{31}^{\otimes 5} \cong \text{Ker} \widetilde{Sq}_*^0 \oplus Q_{13}^{\otimes 5}$. By our previous result in [26] that the invariant space $[Q_{13}^{\otimes 5}]^{GL_5}$ is trivial, and we therefore deduce an estimate

$$\dim[P_{\mathcal{A}_2}((\mathbb{P}_{31}^{\otimes 5})^*)]_{GL_5} = \dim[Q_{31}^{\otimes 5}]^{GL_5} \leq \dim[\text{Ker} \widetilde{Sq}_*^0]^{GL_5}.$$

On the other hand, the works by Lin [15] and Chen [7] show that $\text{Ext}_{\mathcal{A}_2}^{5,5+31}(\mathbb{Z}_2, \mathbb{Z}_2) = \langle n_0, h_0^4 h_5 \rangle$. Moreover, the non-zero elements $h_0^4 h_5$ and n_0 are detected by the rank 5 transfer (see Singer [30], Chơn-Hà [8]). Hence, $\dim[P_{\mathcal{A}_2}((\mathbb{P}_{31}^{\otimes 5})^*)]_{GL_5} \geq 2$. Thus, to prove that $[P_{\mathcal{A}_2}((\mathbb{P}_{31}^{\otimes 5})^*)]_{GL_5}$ has dimension 2, we will show that $\dim[\text{Ker} \widetilde{Sq}_*^0]^{GL_5} \leq 2$. But this is immediate from Theorem 1.6 and the following proposition.

Proposition 2.5.4. *The following holds:*

$$\dim[(Q_{31}^{\otimes 5})^{\omega_{(j)}}]^{GL_5} = \begin{cases} 1 & \text{if } j = 1, 2, \\ 0 & \text{if } j = 3. \end{cases}$$

Proof. We prove the proposition for $j = 1, 3$ and leave the rest to the reader. First of all, we consider the space $(Q_{31}^{\otimes 5})^{\omega_{(1)}} = (Q_{31}^{\otimes 5})^{\omega_{(1)}^0} \oplus (Q_{31}^{\otimes 5})^{\omega_{(1)}^{>0}}$, where as shown in the proofs of Theorem 1.6 and Proposition 2.2.1, the readers can notice that $(Q_{31}^{\otimes 5})^{\omega_{(1)}^0}$ is a $\mathbb{Z}_2$ -vector space of dimension 30 with a basis consisting all the classes represented by the following admissible monomials:

$$\begin{array}{lllll} L_1 = x_5^{31}, & L_2 = x_4^{31}, & L_3 = x_3^{31}, & L_4 = x_2^{31}, & L_5 = x_1^{31}, \\ L_6 = x_4 x_5^{30}, & L_7 = x_3 x_5^{30}, & L_8 = x_3 x_4^{30}, & L_9 = x_2 x_5^{30}, & L_{10} = x_2 x_4^{30}, \\ L_{11} = x_2 x_3^{30}, & L_{12} = x_1 x_5^{30}, & L_{13} = x_1 x_4^{30}, & L_{14} = x_1 x_3^{30}, & L_{15} = x_1 x_2^{30}, \\ L_{16} = x_3 x_4^2 x_5^{28}, & L_{17} = x_2 x_4^2 x_5^{28}, & L_{18} = x_2 x_3^2 x_5^{28}, & L_{19} = x_2 x_3^2 x_4^{28}, & L_{20} = x_1 x_4^2 x_5^{28}, \\ L_{21} = x_1 x_3^2 x_5^{28}, & L_{22} = x_1 x_3^2 x_4^{28}, & L_{23} = x_1 x_2^2 x_5^{28}, & L_{24} = x_1 x_2^2 x_4^{28}, & L_{25} = x_1 x_2^2 x_3^{28}, \\ L_{26} = x_2 x_3^2 x_4^2 x_5^{24}, & L_{27} = x_1 x_3^2 x_4^2 x_5^{24}, & L_{28} = x_1 x_2^2 x_4^2 x_5^{24}, & L_{29} = x_1 x_2^2 x_3^2 x_5^{24}, & L_{30} = x_1 x_2^2 x_3^2 x_4^{24} \end{array}$$

and that $(Q_{31}^{\otimes 5})^{\omega_{(1)}^{>0}}$ is 1-dimensional with the basis $\{[L_{31} = x_1 x_2^2 x_3^4 x_4^8 x_5^{16}]\}$. Thus, an admissible monomial basis of $(Q_{31}^{\otimes 5})^{\omega_{(1)}}$ is the set $\{[L_j]_{\omega_{(1)}} = [L_j] : 1 \leq j \leq 31\}$. Then it is not difficult to check that

$$(Q_{31}^{\otimes 5})^{\omega_{(1)}} \cong \Sigma_5(L_1) \oplus \Sigma_5(L_6) \oplus \Sigma_5(L_{16}) \oplus \Sigma_5(L_{26}) \oplus \Sigma_5(L_{31}),$$

where

$$\begin{aligned} \Sigma_5(L_1) &= \langle \{[L_j] : 1 \leq j \leq 5\} \rangle, & \Sigma_5(L_6) &= \langle \{[L_j] : 6 \leq j \leq 15\} \rangle \\ \Sigma_5(L_{16}) &= \langle \{[L_j] : 16 \leq j \leq 25\} \rangle, & \Sigma_5(L_{26}) &= \langle \{[L_j] : 26 \leq j \leq 30\} \rangle, \\ \Sigma_5(L_{31}) &= \langle [L_{31}] \rangle. \end{aligned}$$

Then, by using the homomorphisms $\sigma_d : \mathbb{P}^{\otimes 5} \rightarrow \mathbb{P}^{\otimes 5}$ for $1 \leq d \leq 4$ and similar computations as in the proof of Proposition 2.5.3, we obtain

$$\begin{aligned} [\Sigma_5(L_1)]^{\Sigma_5} &= \langle [\sum_{1 \leq j \leq 5} L_j] \rangle, \quad [\Sigma_5(L_6)]^{\Sigma_5} = \langle [\sum_{6 \leq j \leq 15} L_j] \rangle, \\ [\Sigma_5(L_{16})]^{\Sigma_5} &= \langle [\sum_{16 \leq j \leq 25} L_j] \rangle, \quad [\Sigma_5(L_{26})]^{\Sigma_5} = \langle [\sum_{26 \leq j \leq 30} L_j] \rangle, \\ [\Sigma_5(L_{31})]^{\Sigma_5} &= \langle [L_{31}] \rangle. \end{aligned}$$

These show that if $[T] \in [(Q_{31}^{\otimes 5})^{\omega(1)}]^{GL_5}$ then

$$T \equiv \gamma_1 \sum_{1 \leq j \leq 5} L_j + \gamma_2 \sum_{6 \leq j \leq 15} L_j + \gamma_3 \sum_{16 \leq j \leq 25} L_j + \gamma_4 \sum_{26 \leq j \leq 30} L_j + \gamma_5 L_{31}$$

where $\gamma_j \in \mathbb{Z}_2$, $1 \leq j \leq 5$. Acting the homomorphism $\sigma_5 : \mathbb{P}^{\otimes 5} \rightarrow \mathbb{P}^{\otimes 5}$ on both sides of this equality and using the relation $\sigma_5(T) + T \equiv 0$, we get

$$(\gamma_1 + \gamma_2)L_4 + (\gamma_2 + \gamma_3)(L_9 + L_{10} + L_{11}) + (\gamma_3 + \gamma_4)(L_{17} + L_{18} + L_{19}) + (\gamma_4 + \gamma_5)L_{26} \equiv 0$$

and therefore $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5$, that is $[(Q_{31}^{\otimes 5})^{\omega(1)}]^{GL_5} = \langle [\sum_{1 \leq j \leq 31} L_j] \rangle$. Therefore $[(Q_{31}^{\otimes 5})^{\omega(1)}]^{GL_5}$ has dimension 1.

Now, we prove the proposition for $j = 3$. Recall that the set $\{[Y_j]_{\omega(3)} : 1 \leq j \leq 70\}$ is an admissible monomial basis of $(Q_{31}^{\otimes 5})^{\omega(3)} = (Q_{31}^{\otimes 5})^{\omega(1)^{>0}}$. By a simple computation, we find that

$$(Q_{31}^{\otimes 5})^{\omega(3)} \cong \Sigma_5(Y_1) \oplus \Sigma_5(Y_{21}) \oplus \mathbb{U},$$

where

$$\begin{aligned} \Sigma_5(Y_1) &= \langle \{[Y_j]_{\omega(3)} : 1 \leq j \leq 20\} \rangle, \\ \Sigma_5(Y_{21}) &= \langle \{[Y_j]_{\omega(3)} : 21 \leq j \leq 50\} \rangle, \\ \mathbb{U} &= \langle \{[Y_j]_{\omega(3)} : 51 \leq j \leq 70\} \rangle. \end{aligned}$$

As a result, under the action of the group Σ_5 , we obtain:

$$\begin{aligned} [\Sigma_5(Y_1)]^{\Sigma_5} &= \langle [\sum_{1 \leq j \leq 20} Y_j]_{\omega(3)} \rangle, \\ [\Sigma_5(Y_{21})]^{\Sigma_5} &= \langle [\sum_{21 \leq j \leq 50} Y_j]_{\omega(3)} \rangle, \\ [\mathbb{U}]^{\Sigma_5} &= 0. \end{aligned}$$

From these calculations, we find that for any $W \in \mathbb{P}_{31}^{\otimes 5}$ such that $[W]_{\omega(3)} \in [(Q_{31}^{\otimes 5})^{\omega(3)}]^{GL_5}$, then $W \equiv_{\omega(3)} \beta_1 \sum_{1 \leq j \leq 20} Y_j + \beta_2 \sum_{21 \leq j \leq 50} Y_j$ where β_1 and β_2 belong to $\mathbb{Z}_2$. Now, it is not too difficult to check that

$$\sigma_5(W) + W \equiv_{\omega(3)} \beta_1 \left(\sum_{1 \leq j \leq 4} Y_j \right) + \beta_2 (Y_8 + Y_{17} + Y_{18}) + \text{other terms} \equiv_{\omega(3)} 0$$

and so $\beta_1 = \beta_2 = 0$. This means that $[(Q_{31}^{\otimes 5})^{\omega(3)}]^{GL_5}$ is trivial. The proposition follows. $\square$

Thus, as argued above, we claim that the coinvariant $[P_{\mathcal{A}_2}((\mathbb{P}_{31}^{\otimes 5})^*)]_{GL_5}$ is 2-dimensional. This completes the proof of the case $n = 31$. $\square$

Proof of Case $n = 32$. It suffices to show that the spaces of GL_5 -invariants $[(Q_{32}^{\otimes 5})^{\omega(j)}]^{GL_5}$ are trivial, for any j , $1 \leq j \leq 3$. Indeed, we prove this statement for $j = 1$. The others can be obtained by similar computations. Let us recall that from the proof of Theorem 1.12, we have $(Q_{32}^{\otimes 5})^{\omega(1)} \cong (Q_{32}^{\otimes 5})^{(\omega(1))^0} \oplus (Q_{32}^{\otimes 5})^{(\omega(1))^{>0}}$, where $\dim(Q_{32}^{\otimes 5})^{(\omega(1))^0} = 115$ and $\dim(Q_{32}^{\otimes 5})^{(\omega(1))^{>0}} = 9$. A simple computation shows that a basis of $(Q_{32}^{\otimes 5})^{(\omega(1))^0}$ is the set of all classes represented by the admissible monomials $\tilde{Z}_j$ for $1 \leq j \leq 115$ (see Subsect.3.6). We put $A(\omega(1)) = \{\tilde{Z}_j : 1 \leq j \leq 115\}$,

then from the calculations in Theorem 1.12, we see that the set $[A(\omega'_{(1)}) \cup B(\omega'_{(1)}) \cup C(\omega'_{(1)})]_{\omega'_{(1)}} = [A(\omega'_{(1)}) \cup B(\omega'_{(1)}) \cup C(\omega'_{(1)})]$ is an admissible basis of $(Q_{32}^{\otimes 5})^{\omega'_{(1)}}$ and therefore we have a direct summand decomposition of the Σ_5 -modules:

$$(Q_{32}^{\otimes 5})^{\omega'_{(1)}} = \Sigma_5(\widetilde{Z}_1) \oplus \Sigma_5(\widetilde{Z}_{21}) \oplus \Sigma_5(\widetilde{Z}_{31}, \widetilde{Z}_{51}) \oplus \Sigma_5(\widetilde{Z}_{81}, \widetilde{Z}_{101}) \oplus \mathbb{M},$$

where

$$\begin{aligned}\Sigma_5(\widetilde{Z}_1) &= \langle \{[\widetilde{Z}_j] : 1 \leq j \leq 20\} \rangle, \\ \Sigma_5(\widetilde{Z}_{21}) &= \langle \{[\widetilde{Z}_j] : 21 \leq j \leq 30\} \rangle, \\ \Sigma_5(\widetilde{Z}_{31}, \widetilde{Z}_{51}) &= \langle \{[\widetilde{Z}_j] : 31 \leq j \leq 80\} \rangle, \\ \Sigma_5(\widetilde{Z}_{81}, \widetilde{Z}_{101}) &= \langle \{[\widetilde{Z}_j] : 81 \leq j \leq 115\} \rangle, \\ \mathbb{M} &= \langle \{[Z_j] : 1 \leq j \leq 9\} \rangle.\end{aligned}$$

Then, based on the homomorphisms $\sigma_1, \dots, \sigma_4 : \mathbb{P}^{\otimes 5} \rightarrow \mathbb{P}^{\otimes 5}$ and direct calculations, it may be concluded that

$$\begin{aligned}[\Sigma_5(\widetilde{Z}_1)]^{\Sigma_5} &= \langle [\widehat{q}_1] \rangle, \text{ with } \widehat{q}_1 := \sum_{1 \leq j \leq 20} \widetilde{Z}_j, \\ [\Sigma_5(\widetilde{Z}_{21})]^{\Sigma_5} &= \langle [\widehat{q}_2] \rangle, \text{ with } \widehat{q}_2 := \sum_{21 \leq j \leq 30} \widetilde{Z}_j, \\ [\Sigma_5(\widetilde{Z}_{31}, \widetilde{Z}_{51})]^{\Sigma_5} &= \langle [\widehat{q}_3] \rangle, \text{ with } \widehat{q}_3 := \widetilde{Z}_{32} + \widetilde{Z}_{34} + \widetilde{Z}_{37} + \widetilde{Z}_{38} + \widetilde{Z}_{40} + \widetilde{Z}_{43} + \widetilde{Z}_{44} + \sum_{48 \leq j \leq 80} \widetilde{Z}_j, \\ [\Sigma_5(\widetilde{Z}_{81}, \widetilde{Z}_{101})]^{\Sigma_5} &= \langle [\widehat{q}_4 + \widehat{q}_5], [\widehat{q}_5 + \widehat{q}_6] \rangle, \text{ where} \\ \widehat{q}_4 &:= \widetilde{Z}_{82} + \widetilde{Z}_{83} + \widetilde{Z}_{85} + \widetilde{Z}_{86} + \sum_{90 \leq j \leq 95} \widetilde{Z}_j, \\ \widehat{q}_5 &:= \widetilde{Z}_{81} + \widetilde{Z}_{84} + \widetilde{Z}_{87} + \widetilde{Z}_{88} + \widetilde{Z}_{89}, \\ \widehat{q}_6 &:= \sum_{96 \leq j \leq 115} \widetilde{Z}_j, \\ [\mathbb{M}]^{\Sigma_5} &= \langle [\widehat{q}_7] \rangle, \text{ with } \widehat{q}_7 := \sum_{2 \leq j \leq 9} Z_j.\end{aligned}$$

Now let us consider $K \in \mathbb{P}_{32}^{\otimes 5}$ such that $[K] \in [(Q_{32}^{\otimes 5})^{\omega'_{(1)}}]^{GL_5}$. Then, since $\Sigma_5 \subset GL_5$, by the above computations, we have

$$K \equiv \sum_{1 \leq i \leq 3} \xi_i \widehat{q}_i + \xi_4(\widehat{q}_4 + \widehat{q}_5) + \xi_5(\widehat{q}_5 + \widehat{q}_6) + \xi_6 \widehat{q}_7$$

where $\xi_i \in \mathbb{Z}_2$. By direct computations using the relation $\sigma_5(K) \equiv K$, we obtain

$$\xi_1 \widetilde{Z}_7 + (\xi_1 + \xi_3) \widetilde{Z}_{10} + (\xi_1 + \xi_2) \widetilde{Z}_{16} + \xi_4 \widetilde{Z}_{33} + (\xi_3 + \xi_5) \widetilde{Z}_{52} + \xi_6 \widetilde{Z}_8 + \text{other terms} \equiv 0,$$

which leads to $\xi_i = 0$ for all i . This claims the result when $n = 32$, and therefore, the proof of Theorem 1.13 is completed. $\square$

3. Appendix

In this section, we describe the following sets: $(\mathcal{C}_{14}^{\otimes 5})^{\omega_{(j)}}$ for $1 \leq j \leq 3$, $(\mathcal{C}_{31}^{\otimes 5})^{\omega_{(j)}^{\geq 0}}$ for $j = 2, 3$ and $(\mathcal{C}_{32}^{\otimes 5})^{\omega_{(j)}}$ for $1 \leq j \leq 3$. At the same time, we provide an algorithm in MAGMA for verifying the dimensions of $Q_{31}^{\otimes 5}$ and $Q_{32}^{\otimes 5}$.

3.1. Admissible monomials in $(\mathcal{C}_{14}^{\otimes 5})^{\widetilde{\omega}_{(1)}}$

$$\begin{aligned}
W_1 &= x_4^7 x_5^7, & W_2 &= x_3^7 x_5^7, & W_3 &= x_3^7 x_4^7, & W_4 &= x_2^7 x_5^7, \\
W_5 &= x_2^7 x_4^7, & W_6 &= x_2^7 x_3^7, & W_7 &= x_1^7 x_5^7, & W_8 &= x_1^7 x_4^7, \\
W_9 &= x_1^7 x_3^7, & W_{10} &= x_1^7 x_2^7, & W_{11} &= x_3 x_4^6 x_5^7, & W_{12} &= x_3 x_4^7 x_5^6, \\
W_{13} &= x_3^7 x_4 x_5^6, & W_{14} &= x_2 x_4^6 x_5^7, & W_{15} &= x_2 x_4^7 x_5^6, & W_{16} &= x_2 x_3^6 x_5^7, \\
W_{17} &= x_2 x_3^6 x_4^7, & W_{18} &= x_2 x_3^7 x_5^6, & W_{19} &= x_2 x_3^7 x_4^6, & W_{20} &= x_2^7 x_4 x_5^6, \\
W_{21} &= x_2^7 x_3 x_5^6, & W_{22} &= x_2^7 x_3 x_4^6, & W_{23} &= x_1 x_4^6 x_5^7, & W_{24} &= x_1 x_4^7 x_5^6, \\
W_{25} &= x_1 x_3^6 x_5^7, & W_{26} &= x_1 x_3^6 x_4^7, & W_{27} &= x_1 x_3^7 x_5^6, & W_{28} &= x_1 x_3^7 x_4^6, \\
W_{29} &= x_1 x_2^6 x_5^7, & W_{30} &= x_1 x_2^6 x_4^7, & W_{31} &= x_1 x_2^6 x_3^7, & W_{32} &= x_1 x_2^7 x_5^6, \\
W_{33} &= x_1 x_2^7 x_4^6, & W_{34} &= x_1 x_2^7 x_3^6, & W_{35} &= x_1^7 x_4 x_5^6, & W_{36} &= x_1^7 x_3 x_5^6, \\
W_{37} &= x_1^7 x_3 x_4^6, & W_{38} &= x_1^7 x_2 x_5^6, & W_{39} &= x_1^7 x_2 x_4^6, & W_{40} &= x_1^7 x_2 x_3^6, \\
W_{41} &= x_3^3 x_4^5 x_5^6, & W_{42} &= x_2^3 x_4^5 x_5^6, & W_{43} &= x_2^3 x_5^5 x_5^6, & W_{44} &= x_2^3 x_3^5 x_5^6, \\
W_{45} &= x_1^3 x_4^5 x_5^6, & W_{46} &= x_1^3 x_3^5 x_5^6, & W_{47} &= x_1^3 x_3^5 x_4^6, & W_{48} &= x_1^3 x_2^5 x_5^6, \\
W_{49} &= x_1^3 x_2^5 x_4^6, & W_{50} &= x_1^3 x_2^5 x_3^6, & W_{51} &= x_2 x_3^2 x_4^7 x_5^7, & W_{52} &= x_2 x_3^2 x_4^7 x_5^4, \\
W_{53} &= x_2 x_3^2 x_4^7 x_5^4, & W_{54} &= x_2 x_3^2 x_4^7 x_5^4, & W_{55} &= x_1 x_3^2 x_4^7 x_5^7, & W_{56} &= x_1 x_3^2 x_4^7 x_5^4, \\
W_{57} &= x_1 x_2^7 x_4^7 x_5^4, & W_{58} &= x_1 x_2^7 x_4^7 x_5^4, & W_{59} &= x_1 x_2^7 x_4^7 x_5^4, & W_{60} &= x_1 x_2^7 x_4^7 x_5^4, \\
W_{61} &= x_1 x_2^7 x_3^7 x_5^4, & W_{62} &= x_1 x_2^7 x_3^7 x_5^4, & W_{63} &= x_1 x_2^7 x_3^7 x_5^4, & W_{64} &= x_1 x_2^7 x_3^7 x_5^4, \\
W_{65} &= x_1 x_2^7 x_3^7 x_5^4, & W_{66} &= x_1 x_2^7 x_3^7 x_5^4, & W_{67} &= x_1^7 x_3 x_2^7 x_5^4, & W_{68} &= x_1^7 x_2 x_2^7 x_5^4, \\
W_{69} &= x_1^7 x_2 x_3^7 x_5^4, & W_{70} &= x_1^7 x_2 x_3^7 x_5^4, & W_{71} &= x_2 x_3 x_4^6 x_5^6, & W_{72} &= x_2 x_3^6 x_4 x_5^6, \\
W_{73} &= x_1 x_3 x_4^6 x_5^6, & W_{74} &= x_1 x_3^6 x_4 x_5^6, & W_{75} &= x_1 x_2 x_4^6 x_5^6, & W_{76} &= x_1 x_2 x_3^6 x_5^6, \\
W_{77} &= x_1 x_2 x_3^6 x_4^6, & W_{78} &= x_1 x_2^6 x_4 x_5^6, & W_{79} &= x_1 x_2^6 x_3 x_5^6, & W_{80} &= x_1 x_2^6 x_3 x_4^6, \\
W_{81} &= x_2 x_3^2 x_4^5 x_5^6, & W_{82} &= x_1 x_3^2 x_4^5 x_5^6, & W_{83} &= x_1 x_2^2 x_4^5 x_5^6, & W_{84} &= x_1 x_2^2 x_3^5 x_5^6, \\
W_{85} &= x_1 x_2^2 x_3^5 x_5^6, & W_{86} &= x_2 x_3^3 x_4^6 x_5^6, & W_{87} &= x_2 x_3^3 x_4^6 x_5^4, & W_{88} &= x_2^3 x_3 x_4^6 x_5^6, \\
W_{89} &= x_2^3 x_3 x_4^6 x_5^4, & W_{90} &= x_1 x_3^3 x_4^6 x_5^6, & W_{91} &= x_1 x_3^3 x_4^6 x_5^4, & W_{92} &= x_1 x_2^3 x_4^6 x_5^6, \\
W_{93} &= x_1 x_2^3 x_4^6 x_5^4, & W_{94} &= x_1 x_2^3 x_4^6 x_5^4, & W_{95} &= x_1 x_2^3 x_4^6 x_5^4, & W_{96} &= x_1 x_2^3 x_4^6 x_5^4, \\
W_{97} &= x_1 x_2^3 x_4^6 x_5^4, & W_{98} &= x_1^3 x_3 x_4^6 x_5^6, & W_{99} &= x_1^3 x_3 x_4^6 x_5^4, & W_{100} &= x_1^3 x_2 x_4^6 x_5^6, \\
W_{101} &= x_1^3 x_2 x_4^6 x_5^4, & W_{102} &= x_1^3 x_2 x_4^6 x_5^4, & W_{103} &= x_1^3 x_2 x_3^4 x_5^6, & W_{104} &= x_1^3 x_2 x_3^4 x_5^4, \\
W_{105} &= x_1^3 x_2 x_3^4 x_5^6, & W_{106} &= x_2^3 x_3^5 x_4^4 x_5^6, & W_{107} &= x_1^3 x_3^5 x_4^4 x_5^6, & W_{108} &= x_1^3 x_2^5 x_4^4 x_5^6, \\
W_{109} &= x_1^3 x_2^5 x_3^4 x_5^6, & W_{110} &= x_1^3 x_2^5 x_3^4 x_5^4, & W_{111} &= x_2^3 x_3^3 x_4^4 x_5^6, & W_{112} &= x_1^3 x_3^3 x_4^4 x_5^6, \\
W_{113} &= x_1^3 x_2^3 x_4^4 x_5^6, & W_{114} &= x_1^3 x_2^3 x_4^4 x_5^4, & W_{115} &= x_1^3 x_2^3 x_4^4 x_5^4, & W_{116} &= x_1 x_2 x_2^2 x_4^4 x_5^6, \\
W_{117} &= x_1 x_2 x_2^2 x_4^4 x_5^6, & W_{118} &= x_1 x_2 x_3^6 x_4^2 x_5^4, & W_{119} &= x_1 x_2^2 x_3 x_4^4 x_5^6, & W_{120} &= x_1 x_2^2 x_3 x_4^4 x_5^4, \\
W_{121} &= x_1 x_2^2 x_3 x_4^4 x_5^4, & W_{122} &= x_1 x_2^6 x_3 x_4^2 x_5^4, & W_{123} &= x_1 x_2^2 x_3^5 x_4^2 x_5^4, & W_{124} &= x_1 x_2^2 x_3^3 x_4^4 x_5^4, \\
W_{125} &= x_1 x_2^3 x_3^2 x_4^4 x_5^4, & W_{126} &= x_1 x_2^3 x_3^4 x_4^2 x_5^4, & W_{127} &= x_1^3 x_2 x_3^2 x_4^4 x_5^4, & W_{128} &= x_1^3 x_2 x_3^4 x_4^2 x_5^4, \\
W_{129} &= x_1 x_2^2 x_3^4 x_4^2 x_5^4, & W_{130} &= x_1^3 x_2 x_3^2 x_4^4 x_5^4.
\end{aligned}$$

3.2. Admissible monomials in $(\mathcal{C}_{14}^{\otimes 5})^{\widetilde{\omega}_{(2)}}$

$$\begin{aligned}
W_{131} &= x_1 x_2^2 x_3^2 x_4^7 x_5^2, & W_{132} &= x_1 x_2^2 x_3^2 x_4^7 x_5^2, & W_{133} &= x_1 x_2^2 x_3^7 x_4^2 x_5^2, & W_{134} &= x_1 x_2^7 x_3^2 x_4^2 x_5^2, \\
W_{135} &= x_1^7 x_2 x_3^2 x_4^2 x_5^2, & W_{136} &= x_1 x_2^2 x_3^2 x_4^3 x_5^6, & W_{137} &= x_1 x_2^2 x_3^3 x_4^2 x_5^6, & W_{138} &= x_1 x_2^2 x_3^3 x_4^6 x_5^2, \\
W_{139} &= x_1 x_2^3 x_3^2 x_4^2 x_5^6, & W_{140} &= x_1 x_2^3 x_3^2 x_4^6 x_5^2, & W_{141} &= x_1 x_2^3 x_3^6 x_4^2 x_5^2, & W_{142} &= x_1^3 x_2 x_3^2 x_4^2 x_5^6, \\
W_{143} &= x_1^3 x_2 x_3^2 x_4^6 x_5^2, & W_{144} &= x_1^3 x_2 x_3^6 x_4^2 x_5^2, & W_{145} &= x_1^3 x_2^5 x_3^2 x_4^2 x_5^2.
\end{aligned}$$

3.3. Admissible monomials in $(\mathcal{C}_{14}^{\otimes 5})^{\widetilde{\omega}(3)}$

$$\begin{aligned}
W_{146} &= x_2 x_3^3 x_4^3 x_5^7, & W_{147} &= x_2 x_3^3 x_4^7 x_5^3, & W_{148} &= x_2 x_3^7 x_4^3 x_5^3, & W_{149} &= x_2^3 x_3 x_4^3 x_5^7, \\
W_{150} &= x_3^3 x_3 x_4^7 x_5^3, & W_{151} &= x_2^3 x_3^3 x_4 x_5^7, & W_{152} &= x_2^3 x_3^3 x_4^3 x_5^5, & W_{153} &= x_2^3 x_3^3 x_4^5 x_5^3, \\
W_{154} &= x_2^3 x_3^3 x_4^7 x_5, & W_{155} &= x_2^3 x_3^5 x_4^3 x_5^3, & W_{156} &= x_2^3 x_3^7 x_4 x_5^3, & W_{157} &= x_2^3 x_3^7 x_4^3 x_5, \\
W_{158} &= x_2^7 x_3 x_4^3 x_5^3, & W_{159} &= x_2^7 x_3^3 x_4 x_5^3, & W_{160} &= x_2^7 x_3^3 x_4^3 x_5, & W_{161} &= x_1 x_3^3 x_4^3 x_5^7, \\
W_{162} &= x_1 x_3^3 x_4^7 x_5^3, & W_{163} &= x_1 x_3^7 x_4^3 x_5^3, & W_{164} &= x_1 x_2^3 x_4^3 x_5^7, & W_{165} &= x_1 x_2^3 x_4^7 x_5^3, \\
W_{166} &= x_1 x_2^3 x_3^3 x_5^7, & W_{167} &= x_1 x_2^3 x_3^3 x_4^7, & W_{168} &= x_1 x_2^3 x_3^7 x_5^3, & W_{169} &= x_1 x_2^3 x_3^7 x_4^3, \\
W_{170} &= x_1 x_2^7 x_3^3 x_5^3, & W_{171} &= x_1 x_2^7 x_3^3 x_5^3, & W_{172} &= x_1 x_2^7 x_3^3 x_4^3, & W_{173} &= x_1^3 x_3 x_4^3 x_5^7, \\
W_{174} &= x_1^3 x_3 x_4^7 x_5^3, & W_{175} &= x_1^3 x_3^3 x_4 x_5^7, & W_{176} &= x_1^3 x_3^3 x_4^3 x_5^5, & W_{177} &= x_1^3 x_3^3 x_4^5 x_5^3, \\
W_{178} &= x_1^3 x_3^3 x_4^7 x_5, & W_{179} &= x_1^3 x_3^5 x_4^3 x_5^3, & W_{180} &= x_1^3 x_3^7 x_4 x_5^3, & W_{181} &= x_1^3 x_3^7 x_4^3 x_5, \\
W_{182} &= x_1^3 x_2 x_4^7 x_5^3, & W_{183} &= x_1^3 x_2 x_4^7 x_5^3, & W_{184} &= x_1^3 x_2 x_3^3 x_5^7, & W_{185} &= x_1^3 x_2 x_3^3 x_4^7, \\
W_{186} &= x_1^3 x_2 x_3^7 x_5^3, & W_{187} &= x_1^3 x_2 x_3^7 x_4^3, & W_{188} &= x_1^3 x_2^3 x_4 x_5^7, & W_{189} &= x_1^3 x_2^3 x_4^3 x_5^5, \\
W_{190} &= x_1^3 x_2^3 x_4^5 x_5^3, & W_{191} &= x_1^3 x_2^3 x_4^7 x_5^3, & W_{192} &= x_1^3 x_2^3 x_3 x_5^7, & W_{193} &= x_1^3 x_2^3 x_3 x_4^7, \\
W_{194} &= x_1^3 x_2^3 x_3^3 x_5^5, & W_{195} &= x_1^3 x_2^3 x_3^3 x_4^5, & W_{196} &= x_1^3 x_2^3 x_3^5 x_5^3, & W_{197} &= x_1^3 x_2^3 x_3^5 x_4^3, \\
W_{198} &= x_1^3 x_2^3 x_3^7 x_5, & W_{199} &= x_1^3 x_2^3 x_3^7 x_4^3, & W_{200} &= x_1^3 x_2^5 x_4^3 x_5^3, & W_{201} &= x_1^3 x_2^5 x_3^3 x_5^3, \\
W_{202} &= x_1^3 x_2^5 x_3^3 x_4^3, & W_{203} &= x_1^3 x_2^7 x_4 x_5^3, & W_{204} &= x_1^3 x_2^7 x_4^3 x_5, & W_{205} &= x_1^3 x_2^7 x_3 x_5^3, \\
W_{206} &= x_1^3 x_2^7 x_3 x_4^3, & W_{207} &= x_1^3 x_2^7 x_3^3 x_5, & W_{208} &= x_1^3 x_2^7 x_3^3 x_4^3, & W_{209} &= x_1^7 x_3 x_4^3 x_5^3, \\
W_{210} &= x_1^7 x_3 x_4^3 x_5^3, & W_{211} &= x_1^7 x_3^3 x_4 x_5^3, & W_{212} &= x_1^7 x_2 x_4^3 x_5^3, & W_{213} &= x_1^7 x_2 x_3^3 x_5^3, \\
W_{214} &= x_1^7 x_2 x_3^3 x_4^3, & W_{215} &= x_1^7 x_2^3 x_4 x_5^3, & W_{216} &= x_1^7 x_2^3 x_4^3 x_5, & W_{217} &= x_1^7 x_2^3 x_3 x_5^3, \\
W_{218} &= x_1^7 x_2^3 x_3^3 x_4^3, & W_{219} &= x_1^7 x_2^3 x_3^3 x_5, & W_{220} &= x_1^7 x_2^3 x_3^3 x_4^3, & W_{221} &= x_1 x_2 x_3^2 x_4^7 x_5^7, \\
W_{222} &= x_1 x_2 x_3^3 x_4^7 x_5^3, & W_{223} &= x_1 x_2 x_3^3 x_4^7 x_5^3, & W_{224} &= x_1 x_2 x_3^3 x_4^3 x_5^6, & W_{225} &= x_1 x_2 x_3^3 x_4^6 x_5^3, \\
W_{226} &= x_1 x_2 x_3^3 x_4^7 x_5^2, & W_{227} &= x_1 x_2 x_3^6 x_4^3 x_5^3, & W_{228} &= x_1 x_2 x_3^7 x_4^3 x_5^3, & W_{229} &= x_1 x_2 x_3^7 x_4^3 x_5^2, \\
W_{230} &= x_1 x_2^2 x_3 x_4^7 x_5^3, & W_{231} &= x_1 x_2^2 x_3 x_4^7 x_5^3, & W_{232} &= x_1 x_2^2 x_3^3 x_4 x_5^7, & W_{233} &= x_1 x_2^2 x_3^3 x_4^3 x_5^5, \\
W_{234} &= x_1 x_2^2 x_3^3 x_4^5 x_5^3, & W_{235} &= x_1 x_2^2 x_3^3 x_4^7 x_5, & W_{236} &= x_1 x_2^2 x_3^5 x_4^3 x_5^3, & W_{237} &= x_1 x_2^2 x_3^7 x_4 x_5^3, \\
W_{238} &= x_1 x_2^2 x_3^7 x_4^3 x_5, & W_{239} &= x_1 x_2^3 x_3 x_4^7 x_5^3, & W_{240} &= x_1 x_2^3 x_3 x_4^3 x_5^6, & W_{241} &= x_1 x_2^3 x_3 x_4^6 x_5^3, \\
W_{242} &= x_1 x_2^3 x_3 x_4^7 x_5^2, & W_{243} &= x_1 x_2^3 x_3^2 x_4 x_5^7, & W_{244} &= x_1 x_2^3 x_3^2 x_4^3 x_5^5, & W_{245} &= x_1 x_2^3 x_3^2 x_4^5 x_5^3, \\
W_{246} &= x_1 x_2^3 x_3^2 x_4^7 x_5, & W_{247} &= x_1 x_2^3 x_3^3 x_4 x_5^6, & W_{248} &= x_1 x_2^3 x_3^3 x_4^2 x_5^5, & W_{249} &= x_1 x_2^3 x_3^3 x_4^4 x_5^4, \\
W_{250} &= x_1 x_2^3 x_3^3 x_4^3 x_5^3, & W_{251} &= x_1 x_2^3 x_3^3 x_4^5 x_5^2, & W_{252} &= x_1 x_2^3 x_3^3 x_4^6 x_5, & W_{253} &= x_1 x_2^3 x_3^3 x_4^3 x_5^3, \\
W_{254} &= x_1 x_2^3 x_3^5 x_4^3 x_5^3, & W_{255} &= x_1 x_2^3 x_3^5 x_4^2 x_5^2, & W_{256} &= x_1 x_2^3 x_3^6 x_4 x_5^3, & W_{257} &= x_1 x_2^3 x_3^6 x_4^3 x_5, \\
W_{258} &= x_1 x_2^3 x_3^7 x_4 x_5^2, & W_{259} &= x_1 x_2^3 x_3^7 x_4^2 x_5, & W_{260} &= x_1 x_2^6 x_3 x_4^3 x_5^3, & W_{261} &= x_1 x_2^6 x_3^3 x_4 x_5^3, \\
W_{262} &= x_1 x_2^6 x_3^3 x_4^3 x_5, & W_{263} &= x_1 x_2^7 x_3 x_4^2 x_5^3, & W_{264} &= x_1 x_2^7 x_3 x_4^3 x_5^2, & W_{265} &= x_1 x_2^7 x_3^2 x_4 x_5^3, \\
W_{266} &= x_1 x_2^7 x_3^2 x_4^3 x_5, & W_{267} &= x_1 x_2^7 x_3^3 x_4 x_5^2, & W_{268} &= x_1 x_2^7 x_3^3 x_4^2 x_5, & W_{269} &= x_1^3 x_2 x_3 x_4^2 x_5^7, \\
W_{270} &= x_1^3 x_2 x_3 x_4^3 x_5^6, & W_{271} &= x_1^3 x_2 x_3 x_4^6 x_5^3, & W_{272} &= x_1^3 x_2 x_3 x_4^7 x_5^2, & W_{273} &= x_1^3 x_2 x_3^2 x_4 x_5^7, \\
W_{274} &= x_1^3 x_2 x_3^2 x_4^3 x_5^5, & W_{275} &= x_1^3 x_2 x_3^2 x_4^5 x_5^3, & W_{276} &= x_1^3 x_2 x_3^2 x_4^7 x_5, & W_{277} &= x_1^3 x_2 x_3^3 x_4 x_5^6, \\
W_{278} &= x_1^3 x_2 x_3^3 x_4^2 x_5^5, & W_{279} &= x_1^3 x_2 x_3^3 x_4^4 x_5^4, & W_{280} &= x_1^3 x_2 x_3^3 x_4^4 x_5^3, & W_{281} &= x_1^3 x_2 x_3^3 x_4^5 x_5^2, \\
W_{282} &= x_1^3 x_2 x_3^3 x_4^6 x_5, & W_{283} &= x_1^3 x_2 x_3^4 x_4^3 x_5^3, & W_{284} &= x_1^3 x_2 x_3^5 x_4^2 x_5^3, & W_{285} &= x_1^3 x_2 x_3^5 x_4^3 x_5^2, \\
W_{286} &= x_1^3 x_2 x_3^6 x_4 x_5^3, & W_{287} &= x_1^3 x_2 x_3^6 x_4^3 x_5, & W_{288} &= x_1^3 x_2 x_3^7 x_4 x_5^2, & W_{289} &= x_1^3 x_2 x_3^7 x_4^2 x_5, \\
W_{290} &= x_1^3 x_2^3 x_3 x_4 x_5^6, & W_{291} &= x_1^3 x_2^3 x_3 x_4^2 x_5^5, & W_{292} &= x_1^3 x_2^3 x_3 x_4^4 x_5^4, & W_{293} &= x_1^3 x_2^3 x_3 x_4^4 x_5^3, \\
W_{294} &= x_1^3 x_2^3 x_3 x_4^5 x_5^2, & W_{295} &= x_1^3 x_2^3 x_3 x_4^6 x_5, & W_{296} &= x_1^3 x_2^3 x_3^3 x_4 x_5^4, & W_{297} &= x_1^3 x_2^3 x_3^3 x_4^4 x_5, \\
W_{298} &= x_1^3 x_2^3 x_3^4 x_4 x_5^3, & W_{299} &= x_1^3 x_2^3 x_3^4 x_4^3 x_5, & W_{300} &= x_1^3 x_2^3 x_3^5 x_4 x_5^2, & W_{301} &= x_1^3 x_2^3 x_3^5 x_4^2 x_5, \\
W_{302} &= x_1^3 x_2^5 x_3 x_4^2 x_5^3, & W_{303} &= x_1^3 x_2^5 x_3 x_4^3 x_5^2, & W_{304} &= x_1^3 x_2^5 x_3 x_4^3 x_5^3, & W_{305} &= x_1^3 x_2^5 x_3^2 x_4^3 x_5,
\end{aligned}$$

$$\begin{aligned}
W_{306} &= x_1^3 x_2^5 x_3^3 x_4 x_5^2, & W_{307} &= x_1^3 x_2^5 x_3^3 x_4^2 x_5, & W_{308} &= x_1^3 x_2^7 x_3 x_4 x_5^2, & W_{309} &= x_1^3 x_2^7 x_3 x_4^2 x_5, \\
W_{310} &= x_1^7 x_2 x_3 x_4^2 x_5^3, & W_{311} &= x_1^7 x_2 x_3 x_4^3 x_5^2, & W_{312} &= x_1^7 x_2 x_3^2 x_4 x_5^3, & W_{313} &= x_1^7 x_2 x_3^2 x_4^2 x_5, \\
W_{314} &= x_1^7 x_2 x_3^2 x_4 x_5^2, & W_{315} &= x_1^7 x_2 x_3^3 x_4^2 x_5, & W_{316} &= x_1^7 x_2^3 x_3 x_4 x_5^2, & W_{317} &= x_1^7 x_2^3 x_3 x_4^2 x_5, \\
W_{318} &= x_1^3 x_2^4 x_3^3 x_4 x_5^3, & W_{319} &= x_1^3 x_2^4 x_3^3 x_4^2 x_5, & W_{320} &= x_1^3 x_2^4 x_3 x_4^3 x_5^3.
\end{aligned}$$

3.4. Admissible monomials in $(\mathcal{C}_{31}^{\otimes 5})^{\omega_{(2)} > 0}$

$$\begin{aligned}
X_1 &= x_1 x_2^2 x_3 x_4^{12} x_5^{15}, & X_2 &= x_1 x_2^2 x_3 x_4^{15} x_5^{12}, & X_3 &= x_1 x_2^2 x_3^{15} x_4 x_5^{12}, & X_4 &= x_1 x_2^2 x_3 x_4^{13} x_5^{14}, \\
X_5 &= x_1 x_2^2 x_3^{13} x_4 x_5^{14}, & X_6 &= x_1 x_2^2 x_3^{13} x_4^{14} x_5, & X_7 &= x_1 x_2^{14} x_3 x_4 x_5^{14}, & X_8 &= x_1 x_2^{14} x_3 x_4^{14} x_5, \\
X_9 &= x_1 x_2 x_3^{14} x_4 x_5^{14}, & X_{10} &= x_1 x_2 x_3^{14} x_4^{14} x_5, & X_{11} &= x_1 x_2 x_3^2 x_4^{12} x_5^{15}, & X_{12} &= x_1 x_2 x_3^2 x_4^{15} x_5^{12}, \\
X_{13} &= x_1 x_2 x_3^2 x_4^{13} x_5^{14}, & X_{14} &= x_1 x_2^3 x_3^{14} x_4 x_5^{12}, & X_{15} &= x_1 x_2^3 x_3^{12} x_4 x_5^{14}, & X_{16} &= x_1 x_2^3 x_3^{12} x_4^{14} x_5, \\
X_{17} &= x_1 x_2^{15} x_3 x_4 x_5^{12}, & X_{18} &= x_1 x_2 x_3 x_4^{14} x_5^{12}, & X_{19} &= x_1 x_2 x_3^3 x_4^{14} x_5^{12}, & X_{20} &= x_1 x_2 x_3^{15} x_4^2 x_5^{12}, \\
X_{21} &= x_1 x_2 x_3^3 x_4^{14} x_5^{12}, & X_{22} &= x_1 x_2^3 x_3 x_4^{12} x_5^{14}, & X_{23} &= x_1 x_2^3 x_3 x_4^{12} x_5^{14}, & X_{24} &= x_1 x_2^3 x_3^{13} x_4^2 x_5^{12}, \\
X_{25} &= x_1 x_2^3 x_3^5 x_4^{10} x_5^{12}, & X_{26} &= x_1 x_2^{15} x_3 x_4^2 x_5^{12}, & X_{27} &= x_1^3 x_2 x_3^{14} x_4 x_5^{12}, & X_{28} &= x_1^3 x_2 x_3^{12} x_4 x_5^{14}, \\
X_{29} &= x_1^3 x_2 x_3^{12} x_4^{14} x_5, & X_{30} &= x_1^3 x_2 x_3 x_4^{14} x_5^{12}, & X_{31} &= x_1^3 x_2 x_3 x_4^{12} x_5^{14}, & X_{32} &= x_1^3 x_2 x_3^{13} x_4^2 x_5^{12}, \\
X_{33} &= x_1^3 x_2 x_3^5 x_4^{10} x_5^{12}, & X_{34} &= x_1^3 x_2^{13} x_3 x_4^2 x_5^{12}, & X_{35} &= x_1^3 x_2^5 x_3 x_4^{10} x_5^{12}, & X_{36} &= x_1^3 x_2^{13} x_3^2 x_4 x_5^{12}, \\
X_{37} &= x_1^3 x_2^5 x_3^{10} x_4 x_5^{12}, & X_{38} &= x_1^{15} x_2 x_3^2 x_4 x_5^{12}, & X_{39} &= x_1^{15} x_2 x_3 x_4^2 x_5^{12}, & X_{40} &= x_1 x_2^2 x_3^4 x_4^9 x_5^{15}, \\
X_{41} &= x_1 x_2^2 x_3^4 x_4^{15} x_5^9, & X_{42} &= x_1 x_2^2 x_3^5 x_4^8 x_5^{15}, & X_{43} &= x_1 x_2^2 x_3^5 x_4^{15} x_5^8, & X_{44} &= x_1 x_2^2 x_3^{12} x_4 x_5^{15}, \\
X_{45} &= x_1 x_2^2 x_3^{12} x_4^{15} x_5, & X_{46} &= x_1 x_2^2 x_3^{15} x_4^9 x_5^9, & X_{47} &= x_1 x_2^2 x_3^{15} x_4^5 x_5^8, & X_{48} &= x_1 x_2^2 x_3^{15} x_4^{12} x_5, \\
X_{49} &= x_1 x_2^2 x_3^4 x_4^8 x_5^{15}, & X_{50} &= x_1 x_2^2 x_3^4 x_4^8 x_5^{15}, & X_{51} &= x_1 x_2^3 x_3^{14} x_4^{12} x_5, & X_{52} &= x_1 x_2^3 x_3^{15} x_4^2 x_5^8, \\
X_{53} &= x_1 x_2^{15} x_3^2 x_4^9 x_5^9, & X_{54} &= x_1 x_2^{15} x_3^2 x_4^5 x_5^8, & X_{55} &= x_1 x_2^{15} x_3^2 x_4^{12} x_5, & X_{56} &= x_1 x_2^{15} x_3^3 x_4^8 x_5^8, \\
X_{57} &= x_1^3 x_2 x_3^4 x_4^8 x_5^{15}, & X_{58} &= x_1^3 x_2 x_3^4 x_4^{15} x_5^8, & X_{59} &= x_1^3 x_2 x_3^{14} x_4^{12} x_5, & X_{60} &= x_1^3 x_2 x_3^{15} x_4^2 x_5^8, \\
X_{61} &= x_1^3 x_2^5 x_3^{10} x_4^{12} x_5, & X_{62} &= x_1^3 x_2^{13} x_3 x_4^{12} x_5, & X_{63} &= x_1^3 x_2^{15} x_3 x_4^8 x_5^8, & X_{64} &= x_1^{15} x_2 x_3^2 x_4^9 x_5^9, \\
X_{65} &= x_1^{15} x_2 x_3^2 x_4^5 x_5^8, & X_{66} &= x_1^{15} x_2 x_3^2 x_4^{12} x_5, & X_{67} &= x_1^{15} x_2 x_3^3 x_4^8 x_5^8, & X_{68} &= x_1^{15} x_2 x_3^3 x_4^8 x_5^8, \\
X_{69} &= x_1 x_2^2 x_3 x_4^{14} x_5^{13}, & X_{70} &= x_1 x_2^2 x_3^{13} x_4^2 x_5^{13}, & X_{71} &= x_1 x_2^2 x_3^3 x_4^{12} x_5^{13}, & X_{72} &= x_1 x_2^2 x_3^3 x_4^{13} x_5^{12}, \\
X_{73} &= x_1 x_2^2 x_3^{13} x_4^2 x_5^{12}, & X_{74} &= x_1 x_2^2 x_3^4 x_4^{11} x_5^{13}, & X_{75} &= x_1 x_2^2 x_3^5 x_4^{10} x_5^{13}, & X_{76} &= x_1 x_2^2 x_3^5 x_4^{11} x_5^{12}, \\
X_{77} &= x_1 x_2^2 x_3^7 x_4^9 x_5^{12}, & X_{78} &= x_1 x_2^2 x_3^7 x_4^8 x_5^{13}, & X_{79} &= x_1 x_2^6 x_3 x_4^{10} x_5^{13}, & X_{80} &= x_1 x_2^6 x_3 x_4^{11} x_5^{12}, \\
X_{81} &= x_1 x_2^6 x_3^{11} x_4 x_5^{12}, & X_{82} &= x_1 x_2^6 x_3^9 x_4^9 x_5^{12}, & X_{83} &= x_1 x_2^6 x_3^8 x_4^{13} x_5, & X_{84} &= x_1 x_2^6 x_3^8 x_4^8 x_5^9, \\
X_{85} &= x_1 x_2^6 x_3^7 x_4^9 x_5^8, & X_{86} &= x_1 x_2^{14} x_3 x_4^2 x_5^{13}, & X_{87} &= x_1 x_2^{14} x_3 x_4^3 x_5^{12}, & X_{88} &= x_1 x_2^{14} x_3^3 x_4 x_5^{12}, \\
X_{89} &= x_1 x_2 x_3^{14} x_4^2 x_5^{13}, & X_{90} &= x_1 x_2 x_3^{14} x_4^3 x_5^{12}, & X_{91} &= x_1 x_2 x_3^2 x_4^{14} x_5^{13}, & X_{92} &= x_1 x_2 x_3^6 x_4^{10} x_5^{13}, \\
X_{93} &= x_1 x_2 x_3^6 x_4^{11} x_5^{12}, & X_{94} &= x_1 x_2 x_3^{12} x_4^2 x_5^{13}, & X_{95} &= x_1 x_2 x_3^2 x_4^{12} x_5^{13}, & X_{96} &= x_1 x_2 x_3^2 x_4^{13} x_5^{12}, \\
X_{97} &= x_1 x_2^3 x_3^{12} x_4^8 x_5^{12}, & X_{98} &= x_1 x_2^3 x_3^4 x_4^{10} x_5^{13}, & X_{99} &= x_1 x_2^3 x_3^4 x_4^{11} x_5^{12}, & X_{100} &= x_1 x_2^3 x_3^6 x_4^9 x_5^{12}, \\
X_{101} &= x_1 x_2^3 x_3^6 x_4^8 x_5^{13}, & X_{102} &= x_1 x_2^3 x_3^{10} x_4 x_5^{12}, & X_{103} &= x_1 x_2^7 x_3^2 x_4^9 x_5^{12}, & X_{104} &= x_1 x_2^7 x_3^2 x_4^8 x_5^{13}, \\
X_{105} &= x_1 x_2^7 x_3^6 x_4^9 x_5^9, & X_{106} &= x_1 x_2^7 x_3^6 x_4^8 x_5^{12}, & X_{107} &= x_1 x_2 x_3^7 x_4^{10} x_5^{12}, & X_{108} &= x_1 x_2^3 x_3^3 x_4^{12} x_5^{12}, \\
X_{109} &= x_1 x_2^3 x_3^7 x_4^8 x_5^{12}, & X_{110} &= x_1 x_2^7 x_3 x_4^{10} x_5^{12}, & X_{111} &= x_1 x_2^7 x_3^3 x_4^8 x_5^{12}, & X_{112} &= x_1 x_2^7 x_3^7 x_4^8 x_5^8, \\
X_{113} &= x_1^3 x_2 x_3^{12} x_4^2 x_5^{13}, & X_{114} &= x_1^3 x_2 x_3^2 x_4^{12} x_5^{13}, & X_{115} &= x_1^3 x_2 x_3^2 x_4^{13} x_5^{12}, & X_{116} &= x_1^3 x_2 x_3^{12} x_4^3 x_5^{12}, \\
X_{117} &= x_1^3 x_2 x_3^4 x_4^{10} x_5^{13}, & X_{118} &= x_1^3 x_2 x_3^4 x_4^{11} x_5^{12}, & X_{119} &= x_1^3 x_2 x_3^6 x_4^9 x_5^{12}, & X_{120} &= x_1^3 x_2 x_3^6 x_4^8 x_5^{13}, \\
X_{121} &= x_1^3 x_2 x_3^3 x_4^{12} x_5^{12}, & X_{122} &= x_1^3 x_2 x_3^7 x_4^8 x_5^{12}, & X_{123} &= x_1^3 x_2^3 x_3 x_4^{12} x_5^{12}, & X_{124} &= x_1^3 x_2^4 x_3 x_4^{10} x_5^{13}, \\
X_{125} &= x_1^3 x_2^7 x_3 x_4^8 x_5^{12}, & X_{126} &= x_1^3 x_2^4 x_3 x_4^{11} x_5^{12}, & X_{127} &= x_1^3 x_2^3 x_3^{12} x_4 x_5^{12}, & X_{128} &= x_1^3 x_2^4 x_3^{11} x_4 x_5^{12}, \\
X_{129} &= x_1^3 x_2^8 x_3 x_4^8 x_5^{12}, & X_{130} &= x_1^3 x_2^5 x_3^2 x_4^9 x_5^{13}, & X_{131} &= x_1^3 x_2^5 x_3^2 x_4^8 x_5^{13}, & X_{132} &= x_1^3 x_2^5 x_3^6 x_4^8 x_5^9, \\
X_{133} &= x_1^3 x_2^5 x_3^6 x_4^9 x_5^8, & X_{134} &= x_1^3 x_2^5 x_3^3 x_4^8 x_5^{12}, & X_{135} &= x_1^3 x_2^5 x_3^7 x_4^8 x_5^8, & X_{136} &= x_1^3 x_2^5 x_3^5 x_4^8 x_5^{12}, \\
X_{137} &= x_1^3 x_2^7 x_3^5 x_4^8 x_5^8, & X_{138} &= x_1^3 x_2^7 x_3^4 x_4^9 x_5^{12}, & X_{139} &= x_1^3 x_2^4 x_3^3 x_4^9 x_5^{12}, & X_{140} &= x_1^3 x_2^3 x_3^4 x_4^8 x_5^{13}, \\
X_{141} &= x_1^7 x_2 x_3^{10} x_4 x_5^{12}, & X_{142} &= x_1^7 x_2 x_3^2 x_4^9 x_5^{12}, & X_{143} &= x_1^7 x_2 x_3^2 x_4^8 x_5^{13}, & X_{144} &= x_1^7 x_2 x_3^6 x_4^8 x_5^9, \\
X_{145} &= x_1^7 x_2 x_3^6 x_4^9 x_5^8, & X_{146} &= x_1^7 x_2 x_3 x_4^{10} x_5^{12}, & X_{147} &= x_1^7 x_2 x_3^3 x_4^8 x_5^{12}, & X_{148} &= x_1^7 x_2 x_3^7 x_4^8 x_5^8, \\
X_{149} &= x_1^7 x_2^3 x_3 x_4^8 x_5^{12}, & X_{150} &= x_1^7 x_2^7 x_3 x_4^8 x_5^8, & X_{151} &= x_1^7 x_2^3 x_3^8 x_4 x_5^{12}, & X_{152} &= x_1^7 x_2^7 x_3^8 x_4 x_5^8, \\
X_{153} &= x_1^7 x_2^3 x_3^5 x_4^8 x_5^8, & X_{154} &= x_1^7 x_2^3 x_3^{10} x_4^5 x_5^8, & X_{155} &= x_1^7 x_2^3 x_3^{12} x_4^3 x_5^{13}, & X_{156} &= x_1^7 x_2^3 x_3^{10} x_4^4 x_5^9, \\
X_{157} &= x_1^7 x_2^2 x_3^9 x_4^9 x_5^{14}, & X_{158} &= x_1^7 x_2^2 x_3^4 x_4^{14} x_5^9, & X_{159} &= x_1^7 x_2^3 x_3^6 x_4^{13} x_5^8, & X_{160} &= x_1^7 x_2^2 x_3^4 x_4^{13} x_5^8, \\
X_{161} &= x_1^7 x_2^2 x_3^7 x_4^{13} x_5^8, & X_{162} &= x_1^7 x_2^2 x_3^7 x_4^{12} x_5^9, & X_{163} &= x_1^7 x_2^2 x_3^7 x_4^{12} x_5^9, & X_{164} &= x_1^7 x_2^2 x_3^6 x_4^{12} x_5^9.
\end{aligned}$$

$$\begin{aligned}
X_{165} &= x_1 x_2^7 x_3^{11} x_4^8 x_5^8, & X_{166} &= x_1 x_2^3 x_3^4 x_4^9 x_5^{14}, & X_{167} &= x_1 x_2^3 x_3^4 x_4^{14} x_5^9, & X_{168} &= x_1 x_2^3 x_3^{14} x_4^4 x_5^9, \\
X_{169} &= x_1 x_2^3 x_3^7 x_4^{12} x_5^8, & X_{170} &= x_1 x_2^7 x_3^3 x_4^{12} x_5^8, & X_{171} &= x_1 x_2^3 x_3^5 x_4^8 x_5^{14}, & X_{172} &= x_1 x_2^3 x_3^5 x_4^{14} x_5^8, \\
X_{173} &= x_1 x_2^3 x_3^{14} x_4^5 x_5^8, & X_{174} &= x_1^3 x_2^5 x_3^{10} x_4^4 x_5^9, & X_{175} &= x_1^3 x_2^5 x_3^{10} x_4^5 x_5^8, & X_{176} &= x_1^3 x_2^5 x_3^2 x_4^{12} x_5^9, \\
X_{177} &= x_1^3 x_2^{13} x_3^2 x_4^4 x_5^9, & X_{178} &= x_1^3 x_2^5 x_3^{11} x_4^4 x_5^8, & X_{179} &= x_1^3 x_2^6 x_3^6 x_4^{13} x_5^8, & X_{180} &= x_1^3 x_2^6 x_3^6 x_4^{12} x_5^9, \\
X_{181} &= x_1^3 x_2^5 x_3^2 x_4^{13} x_5^8, & X_{182} &= x_1^3 x_2^{13} x_3^2 x_4^5 x_5^8, & X_{183} &= x_1^3 x_2^3 x_3^{12} x_4^4 x_5^9, & X_{184} &= x_1^3 x_2^3 x_3^4 x_4^{12} x_5^9, \\
X_{185} &= x_1^3 x_2^5 x_3^3 x_4^{12} x_5^8, & X_{186} &= x_1^3 x_2^3 x_3^5 x_4^{12} x_5^8, & X_{187} &= x_1^3 x_2^3 x_3^{12} x_4^5 x_5^8, & X_{188} &= x_1^3 x_2^{13} x_3^3 x_4^4 x_5^8, \\
X_{189} &= x_1^3 x_2^3 x_3^{13} x_4^4 x_5^8, & X_{190} &= x_1^3 x_2^3 x_3^4 x_4^{13} x_5^8, & X_{191} &= x_1^3 x_2^7 x_3^9 x_4^4 x_5^8, & X_{192} &= x_1^3 x_2^7 x_3^4 x_4^9 x_5^8, \\
X_{193} &= x_1^3 x_2^7 x_3^4 x_4^8 x_5^9, & X_{194} &= x_1^3 x_2^3 x_3^4 x_4^9 x_5^{14}, & X_{195} &= x_1^3 x_2^3 x_3^4 x_4^{14} x_5^9, & X_{196} &= x_1^3 x_2^3 x_3^{14} x_4^4 x_5^9, \\
X_{197} &= x_1^3 x_2^3 x_3^7 x_4^{12} x_5^8, & X_{198} &= x_1^3 x_2^7 x_3^3 x_4^{12} x_5^8, & X_{199} &= x_1^3 x_2^3 x_3^{14} x_4^5 x_5^8, & X_{200} &= x_1^3 x_2^3 x_3^5 x_4^8 x_5^{14}, \\
X_{201} &= x_1^3 x_2^3 x_3^5 x_4^{14} x_5^8, & X_{202} &= x_1^7 x_2^3 x_3^{10} x_4^5 x_5^8, & X_{203} &= x_1^7 x_2^9 x_3^2 x_4^5 x_5^8, & X_{204} &= x_1^7 x_2^9 x_3^2 x_4^4 x_5^9, \\
X_{205} &= x_1^7 x_2^3 x_3^{10} x_4^4 x_5^9, & X_{206} &= x_1^7 x_2^3 x_3^2 x_4^{13} x_5^8, & X_{207} &= x_1^7 x_2^3 x_3^2 x_4^{12} x_5^9, & X_{208} &= x_1^7 x_2^3 x_3^{11} x_4^4 x_5^8, \\
X_{209} &= x_1^7 x_2^{11} x_3^3 x_4^4 x_5^8, & X_{210} &= x_1^7 x_2^3 x_3^9 x_4^4 x_5^8, & X_{211} &= x_1^7 x_2^3 x_3^3 x_4^4 x_5^8, & X_{212} &= x_1^7 x_2^3 x_3^4 x_4^9 x_5^8, \\
X_{213} &= x_1^7 x_2^3 x_3^4 x_4^8 x_5^9, & X_{214} &= x_1^7 x_2^3 x_3^3 x_4^{12} x_5^8, & X_{215} &= x_1^7 x_2^3 x_3^3 x_4^{12} x_5^8.
\end{aligned}$$

3.5. Admissible monomials in $(\mathcal{C}_{31}^{\otimes 5})^{\omega_{(3)}^{>0}}$

$$\begin{aligned}
Y_1 &= x_1 x_2^3 x_3^6 x_4^6 x_5^{15}, & Y_2 &= x_1 x_2^3 x_3^6 x_4^{15} x_5^6, & Y_3 &= x_1 x_2^3 x_3^{15} x_4^6 x_5^6, & Y_4 &= x_1 x_2^{15} x_3^3 x_4^6 x_5^6, \\
Y_5 &= x_1^3 x_2^3 x_3^6 x_4^{15} x_5^6, & Y_6 &= x_1^3 x_2^3 x_3^6 x_4^{15} x_5^6, & Y_7 &= x_1^3 x_2^3 x_3^{15} x_4^6 x_5^6, & Y_8 &= x_1^3 x_2^{15} x_3^3 x_4^6 x_5^6, \\
Y_9 &= x_1^{15} x_2^3 x_3^6 x_4^6 x_5^6, & Y_{10} &= x_1^{15} x_2^3 x_3^6 x_4^6 x_5^6, & Y_{11} &= x_1^3 x_2^5 x_3^2 x_4^6 x_5^{15}, & Y_{12} &= x_1^3 x_2^5 x_3^2 x_4^{15} x_5^6, \\
Y_{13} &= x_1^3 x_2^5 x_3^6 x_4^{15} x_5^6, & Y_{14} &= x_1^3 x_2^5 x_3^6 x_4^{15} x_5^6, & Y_{15} &= x_1^3 x_2^5 x_3^{15} x_4^2 x_5^6, & Y_{16} &= x_1^3 x_2^5 x_3^3 x_4^2 x_5^6, \\
Y_{17} &= x_1^3 x_2^{15} x_3^5 x_4^2 x_5^6, & Y_{18} &= x_1^3 x_2^{15} x_3^5 x_4^2 x_5^6, & Y_{19} &= x_1^{15} x_2^3 x_3^5 x_4^2 x_5^6, & Y_{20} &= x_1^{15} x_2^3 x_3^5 x_4^2 x_5^6, \\
Y_{21} &= x_1 x_2^3 x_3^6 x_4^7 x_5^{14}, & Y_{22} &= x_1 x_2^3 x_3^7 x_4^6 x_5^{14}, & Y_{23} &= x_1 x_2^3 x_3^7 x_4^{14} x_5^6, & Y_{24} &= x_1 x_2^7 x_3^3 x_4^6 x_5^{14}, \\
Y_{25} &= x_1 x_2^7 x_3^3 x_4^{14} x_5^6, & Y_{26} &= x_1 x_2^3 x_3^6 x_4^7 x_5^{14}, & Y_{27} &= x_1^3 x_2^3 x_3^7 x_4^6 x_5^{14}, & Y_{28} &= x_1^3 x_2^3 x_3^7 x_4^{14} x_5^6, \\
Y_{29} &= x_1^3 x_2^7 x_3^6 x_4^5 x_5^{14}, & Y_{30} &= x_1^3 x_2^7 x_3^3 x_4^{14} x_5^6, & Y_{31} &= x_1^7 x_2^3 x_3^6 x_4^5 x_5^{14}, & Y_{32} &= x_1^7 x_2^3 x_3^3 x_4^{14} x_5^6, \\
Y_{33} &= x_1^7 x_2^3 x_3^6 x_4^5 x_5^{14}, & Y_{34} &= x_1^7 x_2^3 x_3^3 x_4^{14} x_5^6, & Y_{35} &= x_1^3 x_2^5 x_3^2 x_4^7 x_5^{14}, & Y_{36} &= x_1^3 x_2^5 x_3^2 x_4^7 x_5^{14}, \\
Y_{37} &= x_1^3 x_2^5 x_3^7 x_4^{14} x_5^6, & Y_{38} &= x_1^3 x_2^5 x_3^5 x_4^2 x_5^{14}, & Y_{39} &= x_1^3 x_2^7 x_3^5 x_4^{14} x_5^6, & Y_{40} &= x_1^7 x_2^3 x_3^5 x_4^2 x_5^{14}, \\
Y_{41} &= x_1^7 x_2^3 x_3^5 x_4^{14} x_5^6, & Y_{42} &= x_1^3 x_2^7 x_3^{13} x_4^2 x_5^6, & Y_{43} &= x_1^3 x_2^7 x_3^{13} x_4^2 x_5^6, & Y_{44} &= x_1^7 x_2^3 x_3^{13} x_4^2 x_5^6, \\
Y_{45} &= x_1^7 x_2^3 x_3^{13} x_4^2 x_5^6, & Y_{46} &= x_1 x_2^7 x_3^{11} x_4^6 x_5^6, & Y_{47} &= x_1^7 x_2^3 x_3^{11} x_4^6 x_5^6, & Y_{48} &= x_1^7 x_2^{11} x_3^3 x_4^6 x_5^6, \\
Y_{49} &= x_1^7 x_2^{11} x_3^5 x_4^2 x_5^6, & Y_{50} &= x_1^7 x_2^{11} x_3^5 x_4^2 x_5^6, & Y_{51} &= x_1^3 x_2^3 x_3^5 x_4^6 x_5^{14}, & Y_{52} &= x_1^3 x_2^3 x_3^5 x_4^{14} x_5^6, \\
Y_{53} &= x_1^3 x_2^5 x_3^6 x_4^5 x_5^{14}, & Y_{54} &= x_1^3 x_2^5 x_3^3 x_4^{14} x_5^6, & Y_{55} &= x_1^3 x_2^5 x_3^6 x_4^3 x_5^{14}, & Y_{56} &= x_1^3 x_2^5 x_3^6 x_4^{14} x_5^3, \\
Y_{57} &= x_1^3 x_2^3 x_3^{13} x_4^6 x_5^6, & Y_{58} &= x_1^3 x_2^{13} x_3^3 x_4^6 x_5^6, & Y_{59} &= x_1^3 x_2^5 x_3^6 x_4^6 x_5^{11}, & Y_{60} &= x_1^3 x_2^5 x_3^{11} x_4^6 x_5^6, \\
Y_{61} &= x_1^3 x_2^5 x_3^6 x_4^7 x_5^{10}, & Y_{62} &= x_1^3 x_2^5 x_3^7 x_4^6 x_5^{10}, & Y_{63} &= x_1^3 x_2^5 x_3^7 x_4^{10} x_5^6, & Y_{64} &= x_1^3 x_2^7 x_3^3 x_4^6 x_5^{10}, \\
Y_{65} &= x_1^3 x_2^5 x_3^5 x_4^{10} x_5^6, & Y_{66} &= x_1^7 x_2^3 x_3^5 x_4^{10} x_5^6, & Y_{67} &= x_1^7 x_2^3 x_3^5 x_4^{10} x_5^6, & Y_{68} &= x_1^3 x_2^7 x_3^9 x_4^6 x_5^6, \\
Y_{69} &= x_1^7 x_2^3 x_3^9 x_4^6 x_5^6, & Y_{70} &= x_1^7 x_2^3 x_3^6 x_4^5 x_5^6.
\end{aligned}$$

3.6. Admissible monomials in $(\mathcal{C}_{32}^{\otimes 5})^{\omega'_{(1)}}$

We have $(\mathcal{C}_{32}^{\otimes 5})^{\omega'_{(1)}} = (\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(1)})^0} \cup (\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(1)})^{>0}}$, where

$$\begin{aligned}
(\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(1)})^0} &= A(\omega'_{(1)}) = \{\widetilde{Z}_j : 1 \leq j \leq 115\}, \\
(\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(1)})^{>0}} &= B(\omega'_{(1)}) \cup C(\omega'_{(1)}) = \{Z_j : 1 \leq j \leq 9\}.
\end{aligned}$$

The admissible monomials $\widetilde{Z}_j$ and Z_j are determined as follows:

$$\begin{aligned}
\widetilde{Z}_1 &= x_4 x_5^{31}, & \widetilde{Z}_2 &= x_4^3 x_5, & \widetilde{Z}_3 &= x_3 x_5^{31}, & \widetilde{Z}_4 &= x_3 x_4^{31}, \\
\widetilde{Z}_5 &= x_3^3 x_5, & \widetilde{Z}_6 &= x_3^3 x_4, & \widetilde{Z}_7 &= x_2 x_5^{31}, & \widetilde{Z}_8 &= x_2 x_4^{31}, \\
\widetilde{Z}_9 &= x_2 x_3^{31}, & \widetilde{Z}_{10} &= x_2^3 x_5, & \widetilde{Z}_{11} &= x_2^3 x_4, & \widetilde{Z}_{12} &= x_2^3 x_3, \\
\widetilde{Z}_{13} &= x_1 x_5^{31}, & \widetilde{Z}_{14} &= x_1 x_4^{31}, & \widetilde{Z}_{15} &= x_1 x_3^{31}, & \widetilde{Z}_{16} &= x_1 x_2^{31},
\end{aligned}$$

$$\begin{aligned}
\widetilde{Z}_{17} &= x_1^{31} x_5, & \widetilde{Z}_{18} &= x_1^{31} x_4, & \widetilde{Z}_{19} &= x_1^{31} x_3, & \widetilde{Z}_{20} &= x_1^{31} x_2, \\
\widetilde{Z}_{21} &= x_4^3 x_5^{29}, & \widetilde{Z}_{22} &= x_3^3 x_5^{29}, & \widetilde{Z}_{23} &= x_3^3 x_4^{29}, & \widetilde{Z}_{24} &= x_2^3 x_5^{29}, \\
\widetilde{Z}_{25} &= x_2^3 x_4^{29}, & \widetilde{Z}_{26} &= x_2^3 x_3^{29}, & \widetilde{Z}_{27} &= x_1^3 x_5^{29}, & \widetilde{Z}_{28} &= x_1^3 x_4^{29}, \\
\widetilde{Z}_{29} &= x_1^3 x_3^{29}, & \widetilde{Z}_{30} &= x_1^3 x_2^{29}, & \widetilde{Z}_{31} &= x_3 x_4 x_5^{30}, & \widetilde{Z}_{32} &= x_3 x_4^{30} x_5, \\
\widetilde{Z}_{33} &= x_2 x_4 x_5^{30}, & \widetilde{Z}_{34} &= x_2 x_4^{30} x_5, & \widetilde{Z}_{35} &= x_2 x_3 x_5^{30}, & \widetilde{Z}_{36} &= x_2 x_3 x_4^{30}, \\
\widetilde{Z}_{37} &= x_2 x_3^{30} x_5, & \widetilde{Z}_{38} &= x_2 x_3^{30} x_4, & \widetilde{Z}_{39} &= x_1 x_4 x_5^{30}, & \widetilde{Z}_{40} &= x_1 x_4^{30} x_5, \\
\widetilde{Z}_{41} &= x_1 x_3 x_5^{30}, & \widetilde{Z}_{42} &= x_1 x_3 x_4^{30}, & \widetilde{Z}_{43} &= x_1 x_3^{30} x_5, & \widetilde{Z}_{44} &= x_1 x_3^{30} x_4, \\
\widetilde{Z}_{45} &= x_1 x_2 x_5^{30}, & \widetilde{Z}_{46} &= x_1 x_2 x_4^{30}, & \widetilde{Z}_{47} &= x_1 x_2 x_3^{30}, & \widetilde{Z}_{48} &= x_1 x_2^{30} x_5, \\
\widetilde{Z}_{49} &= x_1 x_2^{30} x_4, & \widetilde{Z}_{50} &= x_1 x_2^{30} x_3, & \widetilde{Z}_{51} &= x_3 x_4^2 x_5^{29}, & \widetilde{Z}_{52} &= x_2 x_4^2 x_5^{29}, \\
\widetilde{Z}_{53} &= x_2 x_3^2 x_5^{29}, & \widetilde{Z}_{54} &= x_2 x_3^2 x_4^{29}, & \widetilde{Z}_{55} &= x_1 x_4^2 x_5^{29}, & \widetilde{Z}_{56} &= x_1 x_3^2 x_5^{29}, \\
\widetilde{Z}_{57} &= x_1 x_3^2 x_4^{29}, & \widetilde{Z}_{58} &= x_1 x_2^2 x_5^{29}, & \widetilde{Z}_{59} &= x_1 x_2^2 x_4^{29}, & \widetilde{Z}_{60} &= x_1 x_2^2 x_3^{29}, \\
\widetilde{Z}_{61} &= x_3 x_4^3 x_5^{28}, & \widetilde{Z}_{62} &= x_3^3 x_4 x_5^{28}, & \widetilde{Z}_{63} &= x_2 x_4^3 x_5^{28}, & \widetilde{Z}_{64} &= x_2 x_3^3 x_5^{28}, \\
\widetilde{Z}_{65} &= x_2 x_3^3 x_4^{28}, & \widetilde{Z}_{66} &= x_2^3 x_4 x_5^{28}, & \widetilde{Z}_{67} &= x_2^3 x_3 x_5^{28}, & \widetilde{Z}_{68} &= x_2^3 x_3 x_4^{28}, \\
\widetilde{Z}_{69} &= x_1 x_4^3 x_5^{28}, & \widetilde{Z}_{70} &= x_1 x_3^3 x_5^{28}, & \widetilde{Z}_{71} &= x_1 x_3^3 x_4^{28}, & \widetilde{Z}_{72} &= x_1 x_2^3 x_5^{28}, \\
\widetilde{Z}_{73} &= x_1 x_2^3 x_4^{28}, & \widetilde{Z}_{74} &= x_1 x_2^3 x_3^{28}, & \widetilde{Z}_{75} &= x_1^3 x_4 x_5^{28}, & \widetilde{Z}_{76} &= x_1^3 x_3 x_5^{28}, \\
\widetilde{Z}_{77} &= x_1^3 x_3 x_4^{28}, & \widetilde{Z}_{78} &= x_1^3 x_2 x_5^{28}, & \widetilde{Z}_{79} &= x_1^3 x_2 x_4^{28}, & \widetilde{Z}_{80} &= x_1^3 x_2 x_3^{28}, \\
\widetilde{Z}_{81} &= x_2 x_3 x_4^2 x_5^{28}, & \widetilde{Z}_{82} &= x_2 x_3^2 x_4 x_5^{28}, & \widetilde{Z}_{83} &= x_2 x_3^2 x_4^{28} x_5, & \widetilde{Z}_{84} &= x_1 x_3 x_4^2 x_5^{28}, \\
\widetilde{Z}_{85} &= x_1 x_3^2 x_4 x_5^{28}, & \widetilde{Z}_{86} &= x_1 x_3^2 x_4^{28} x_5, & \widetilde{Z}_{87} &= x_1 x_2 x_4^2 x_5^{28}, & \widetilde{Z}_{88} &= x_1 x_2 x_3^2 x_5^{28}, \\
\widetilde{Z}_{89} &= x_1 x_2 x_3^2 x_4^{28}, & \widetilde{Z}_{90} &= x_1 x_2^2 x_4 x_5^{28}, & \widetilde{Z}_{91} &= x_1 x_2^2 x_4^{28} x_5, & \widetilde{Z}_{92} &= x_1 x_2^2 x_3 x_5^{28}, \\
\widetilde{Z}_{93} &= x_1 x_2^2 x_3 x_4^{28}, & \widetilde{Z}_{94} &= x_1 x_2^2 x_3^{28} x_5, & \widetilde{Z}_{95} &= x_1 x_2^2 x_3^{28} x_4, & \widetilde{Z}_{96} &= x_2 x_3^2 x_4^2 x_5^{25}, \\
\widetilde{Z}_{97} &= x_1 x_3^2 x_4^2 x_5^{25}, & \widetilde{Z}_{98} &= x_1 x_2^2 x_4^2 x_5^{25}, & \widetilde{Z}_{99} &= x_1 x_2^2 x_3^2 x_5^{25}, & \widetilde{Z}_{100} &= x_1 x_2^2 x_3^2 x_4^{25}, \\
\widetilde{Z}_{101} &= x_2 x_3^2 x_4^2 x_5^{24}, & \widetilde{Z}_{102} &= x_1 x_3^2 x_4^2 x_5^{24}, & \widetilde{Z}_{103} &= x_1 x_2^2 x_4^2 x_5^{24}, & \widetilde{Z}_{104} &= x_1 x_2^2 x_3^2 x_5^{24}, \\
\widetilde{Z}_{105} &= x_1 x_2^2 x_3^2 x_4^{24}, & \widetilde{Z}_{106} &= x_2 x_3^2 x_4^2 x_5^{24}, & \widetilde{Z}_{107} &= x_2^3 x_3 x_4^2 x_5^{24}, & \widetilde{Z}_{108} &= x_1 x_3^2 x_4^2 x_5^{24}, \\
\widetilde{Z}_{109} &= x_1 x_2^2 x_3^2 x_4^{24}, & \widetilde{Z}_{110} &= x_1 x_3^2 x_4^2 x_5^{24}, & \widetilde{Z}_{111} &= x_1 x_2^2 x_3^2 x_4^{24}, & \widetilde{Z}_{112} &= x_1^3 x_3 x_4^2 x_5^{24}, \\
\widetilde{Z}_{113} &= x_1^3 x_2 x_4^2 x_5^{24}, & \widetilde{Z}_{114} &= x_1^3 x_2 x_3^2 x_5^{24}, & \widetilde{Z}_{115} &= x_1^3 x_2 x_3^2 x_4^{24}, & Z_1 &= x_1 x_2 x_3^2 x_4^2 x_5^{24}, \\
Z_2 &= x_1 x_2^2 x_3^2 x_4^{24}, & Z_3 &= x_1 x_2^2 x_3^2 x_4 x_5^{24}, & Z_4 &= x_1 x_2^2 x_3^2 x_4^2 x_5^{24}, & Z_5 &= x_1 x_2^2 x_3^2 x_4^2 x_5^{17}, \\
Z_6 &= x_1 x_2^2 x_3^2 x_4^2 x_5^{16}, & Z_7 &= x_1 x_2^2 x_3^2 x_4^2 x_5^{16}, & Z_8 &= x_1 x_2^2 x_3^2 x_4^2 x_5^{16}, & Z_9 &= x_1^3 x_2 x_3^2 x_4^2 x_5^{16}.
\end{aligned}$$

3.7. Admissible monomials in $(\mathcal{C}_{32}^{\otimes 5})^{\omega'_{(2)}}$

We have $(\mathcal{C}_{32}^{\otimes 5})^{\omega'_{(2)}} = (\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(2)})^0} \cup (\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(2)})^{>0}}$, where

$$\begin{aligned}
(\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(2)})^0} &= A(\omega'_{(2)}) = \{\widetilde{Z}'_j : 1 \leq j \leq 175\}, \\
(\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(2)})^{>0}} &= B(\omega'_{(2)}) \cup C(\omega'_{(2)}) = \{Z'_j : 1 \leq j \leq 310\}.
\end{aligned}$$

The admissible monomials $\widetilde{Z}'_j$ and Z'_j are determined as follows:

$$\begin{aligned}
\widetilde{Z}'_1 &= x_2 x_3 x_4^{15} x_5^{15}, & \widetilde{Z}'_2 &= x_2 x_3^3 x_4^{13} x_5^{15}, & \widetilde{Z}'_3 &= x_2 x_3^3 x_4^{15} x_5^{13}, & \widetilde{Z}'_4 &= x_2 x_3^7 x_4^{11} x_5^{13}, \\
\widetilde{Z}'_5 &= x_2 x_3^{15} x_4 x_5^{15}, & \widetilde{Z}'_6 &= x_2 x_3^{15} x_4^3 x_5^{13}, & \widetilde{Z}'_7 &= x_2 x_3^{15} x_4^{15} x_5, & \widetilde{Z}'_8 &= x_2^3 x_3 x_4^{13} x_5^{15}, \\
\widetilde{Z}'_9 &= x_2^3 x_3 x_4^{15} x_5^{13}, & \widetilde{Z}'_{10} &= x_2^3 x_3^3 x_4^{13} x_5^{13}, & \widetilde{Z}'_{11} &= x_2^3 x_3^5 x_4^9 x_5^{15}, & \widetilde{Z}'_{12} &= x_2^3 x_3^5 x_4^{11} x_5^{13}, \\
\widetilde{Z}'_{13} &= x_2^3 x_3^5 x_4^{15} x_5^9, & \widetilde{Z}'_{14} &= x_2^3 x_3^7 x_4^9 x_5^{13}, & \widetilde{Z}'_{15} &= x_2^3 x_3^7 x_4^{13} x_5^9, & \widetilde{Z}'_{16} &= x_2^3 x_3^{13} x_4 x_5^{15}, \\
\widetilde{Z}'_{17} &= x_2^3 x_3^{13} x_4^3 x_5^{13}, & \widetilde{Z}'_{18} &= x_2^3 x_3^{13} x_4^{15} x_5, & \widetilde{Z}'_{19} &= x_2^3 x_3^{15} x_4 x_5^{13}, & \widetilde{Z}'_{20} &= x_2^3 x_3^{15} x_4^5 x_5^9, \\
\widetilde{Z}'_{21} &= x_2^3 x_3^{15} x_4^{13} x_5, & \widetilde{Z}'_{22} &= x_2^7 x_3 x_4^{11} x_5^{13}, & \widetilde{Z}'_{23} &= x_2^7 x_3^3 x_4^9 x_5^{13}, & \widetilde{Z}'_{24} &= x_2^7 x_3^3 x_4^{13} x_5^9, \\
\widetilde{Z}'_{25} &= x_2^7 x_3^7 x_4^9 x_5^9, & \widetilde{Z}'_{26} &= x_2^7 x_3^{11} x_4 x_5^{13}, & \widetilde{Z}'_{27} &= x_2^7 x_3^{11} x_4^5 x_5^9, & \widetilde{Z}'_{28} &= x_2^7 x_3^{11} x_4^{13} x_5, \\
\widetilde{Z}'_{29} &= x_2^{15} x_3 x_4 x_5^{15}, & \widetilde{Z}'_{30} &= x_2^{15} x_3^3 x_4 x_5^{13}, & \widetilde{Z}'_{31} &= x_2^{15} x_3 x_4^{15} x_5, & \widetilde{Z}'_{32} &= x_2^{15} x_3^3 x_4 x_5^{13},
\end{aligned}$$

$$\begin{aligned}
\widetilde{Z'_{33}} &= x_2^{15} x_3^3 x_4^5 x_5^9, & \widetilde{Z'_{34}} &= x_2^{15} x_3^3 x_4^{13} x_5, & \widetilde{Z'_{35}} &= x_2^{15} x_3^{15} x_4 x_5, & \widetilde{Z'_{36}} &= x_1 x_3 x_4^{15} x_5^{15}, \\
\widetilde{Z'_{37}} &= x_1 x_3^3 x_4^{13} x_5^{15}, & \widetilde{Z'_{38}} &= x_1 x_3^3 x_4^{15} x_5^{13}, & \widetilde{Z'_{39}} &= x_1 x_3^7 x_4^{11} x_5^{13}, & \widetilde{Z'_{40}} &= x_1 x_3^{15} x_4 x_5^{15}, \\
\widetilde{Z'_{41}} &= x_1 x_3^{15} x_4^3 x_5^{13}, & \widetilde{Z'_{42}} &= x_1 x_3^{15} x_4^{15} x_5, & \widetilde{Z'_{43}} &= x_1 x_2 x_4^{15} x_5^{15}, & \widetilde{Z'_{44}} &= x_1 x_2 x_3^{15} x_5^{15}, \\
\widetilde{Z'_{45}} &= x_1 x_2 x_3^{15} x_4^{15}, & \widetilde{Z'_{46}} &= x_1 x_2^3 x_4^{13} x_5^{15}, & \widetilde{Z'_{47}} &= x_1 x_2^3 x_4^{15} x_5^{13}, & \widetilde{Z'_{48}} &= x_1 x_2^3 x_3^{13} x_5^{15}, \\
\widetilde{Z'_{49}} &= x_1 x_2^3 x_3^{13} x_4^{15}, & \widetilde{Z'_{50}} &= x_1 x_2^3 x_3^{15} x_5^{13}, & \widetilde{Z'_{51}} &= x_1 x_2^3 x_3^{15} x_4^{13}, & \widetilde{Z'_{52}} &= x_1 x_2^7 x_4^{11} x_5^{13}, \\
\widetilde{Z'_{53}} &= x_1 x_2^7 x_3^{11} x_5^{13}, & \widetilde{Z'_{54}} &= x_1 x_2^7 x_3^{11} x_4^{13}, & \widetilde{Z'_{55}} &= x_1 x_2^{15} x_4 x_5^{15}, & \widetilde{Z'_{56}} &= x_1 x_2^{15} x_4^3 x_5^{13}, \\
\widetilde{Z'_{57}} &= x_1 x_2^{15} x_4^{15} x_5, & \widetilde{Z'_{58}} &= x_1 x_2^{15} x_3 x_5^{15}, & \widetilde{Z'_{59}} &= x_1 x_2^{15} x_3 x_4^{15}, & \widetilde{Z'_{60}} &= x_1 x_2^{15} x_3^3 x_5^{13}, \\
\widetilde{Z'_{61}} &= x_1 x_2^{15} x_3^3 x_4^{13}, & \widetilde{Z'_{62}} &= x_1 x_2^{15} x_3^{15} x_5, & \widetilde{Z'_{63}} &= x_1 x_2^{15} x_3^{15} x_4, & \widetilde{Z'_{64}} &= x_1^3 x_3 x_4^{13} x_5^{15}, \\
\widetilde{Z'_{65}} &= x_1^3 x_3 x_4^{15} x_5^{13}, & \widetilde{Z'_{66}} &= x_1^3 x_3^3 x_4^{13} x_5^{13}, & \widetilde{Z'_{67}} &= x_1^3 x_3^5 x_4^9 x_5^{15}, & \widetilde{Z'_{68}} &= x_1^3 x_3^5 x_4^{11} x_5^{13}, \\
\widetilde{Z'_{69}} &= x_1^3 x_3^5 x_4^{15} x_5^9, & \widetilde{Z'_{70}} &= x_1^3 x_3^7 x_4^9 x_5^{13}, & \widetilde{Z'_{71}} &= x_1^3 x_3^7 x_4^{13} x_5^9, & \widetilde{Z'_{72}} &= x_1^3 x_3^{13} x_4 x_5^{15}, \\
\widetilde{Z'_{73}} &= x_1^3 x_3^{13} x_4^3 x_5^{13}, & \widetilde{Z'_{74}} &= x_1^3 x_3^{13} x_4^{15} x_5, & \widetilde{Z'_{75}} &= x_1^3 x_3^{15} x_4 x_5^{13}, & \widetilde{Z'_{76}} &= x_1^3 x_3^{15} x_4^5 x_5^9, \\
\widetilde{Z'_{77}} &= x_1^3 x_3^{15} x_4^{13} x_5, & \widetilde{Z'_{78}} &= x_1^3 x_2 x_4^{13} x_5^{15}, & \widetilde{Z'_{79}} &= x_1^3 x_2 x_4^{15} x_5^{13}, & \widetilde{Z'_{80}} &= x_1^3 x_2 x_3^{13} x_5^{15}, \\
\widetilde{Z'_{81}} &= x_1^3 x_2 x_3^{13} x_4^{15}, & \widetilde{Z'_{82}} &= x_1^3 x_2 x_3^{15} x_5^{13}, & \widetilde{Z'_{83}} &= x_1^3 x_2 x_3^{15} x_4^{13}, & \widetilde{Z'_{84}} &= x_1^3 x_2^3 x_4^{13} x_5^{13}, \\
\widetilde{Z'_{85}} &= x_1^3 x_2^3 x_3^{13} x_5^{13}, & \widetilde{Z'_{86}} &= x_1^3 x_2^3 x_3^{13} x_4^{13}, & \widetilde{Z'_{87}} &= x_1^3 x_2^5 x_4^9 x_5^{15}, & \widetilde{Z'_{88}} &= x_1^3 x_2^5 x_4^{11} x_5^{13}, \\
\widetilde{Z'_{89}} &= x_1^3 x_2^5 x_4^{15} x_5^9, & \widetilde{Z'_{90}} &= x_1^3 x_2^5 x_3^9 x_5^{15}, & \widetilde{Z'_{91}} &= x_1^3 x_2^5 x_3^9 x_4^{15}, & \widetilde{Z'_{92}} &= x_1^3 x_2^5 x_3^{11} x_5^{13}, \\
\widetilde{Z'_{93}} &= x_1^3 x_2^5 x_3^{11} x_4^{13}, & \widetilde{Z'_{94}} &= x_1^3 x_2^5 x_3^{15} x_5^9, & \widetilde{Z'_{95}} &= x_1^3 x_2^5 x_3^{15} x_4^9, & \widetilde{Z'_{96}} &= x_1^3 x_2^7 x_4^9 x_5^{13}, \\
\widetilde{Z'_{97}} &= x_1^3 x_2^7 x_4^{13} x_5^9, & \widetilde{Z'_{98}} &= x_1^3 x_2^7 x_3^9 x_5^{13}, & \widetilde{Z'_{99}} &= x_1^3 x_2^7 x_3^9 x_4^{13}, & \widetilde{Z'_{100}} &= x_1^3 x_2^7 x_3^{13} x_5^9, \\
\widetilde{Z'_{101}} &= x_1^3 x_2^7 x_3^{13} x_4^9, & \widetilde{Z'_{102}} &= x_1^3 x_2^{13} x_4 x_5^{15}, & \widetilde{Z'_{103}} &= x_1^3 x_2^{13} x_4^3 x_5^{13}, & \widetilde{Z'_{104}} &= x_1^3 x_2^{13} x_4^{15} x_5, \\
\widetilde{Z'_{105}} &= x_1^3 x_2^{13} x_3 x_5^{15}, & \widetilde{Z'_{106}} &= x_1^3 x_2^{13} x_3 x_4^{13}, & \widetilde{Z'_{107}} &= x_1^3 x_2^{13} x_3^3 x_5^{13}, & \widetilde{Z'_{108}} &= x_1^3 x_2^{13} x_3^3 x_4^{13}, \\
\widetilde{Z'_{109}} &= x_1^3 x_2^{13} x_3^{15} x_5, & \widetilde{Z'_{110}} &= x_1^3 x_2^{13} x_3^{15} x_4, & \widetilde{Z'_{111}} &= x_1^3 x_2^{15} x_4 x_5^{13}, & \widetilde{Z'_{112}} &= x_1^3 x_2^{15} x_4^5 x_5^9, \\
\widetilde{Z'_{113}} &= x_1^3 x_2^{15} x_4^{13} x_5, & \widetilde{Z'_{114}} &= x_1^3 x_2^{15} x_3 x_5^{13}, & \widetilde{Z'_{115}} &= x_1^3 x_2^{15} x_3 x_4^{13}, & \widetilde{Z'_{116}} &= x_1^3 x_2^{15} x_3^5 x_5^9, \\
\widetilde{Z'_{117}} &= x_1^3 x_2^{15} x_3^9 x_4^9, & \widetilde{Z'_{118}} &= x_1^3 x_2^{15} x_3^{13} x_5, & \widetilde{Z'_{119}} &= x_1^3 x_2^{15} x_3^{13} x_4, & \widetilde{Z'_{120}} &= x_1^7 x_3 x_4^{11} x_5^{13}, \\
\widetilde{Z'_{121}} &= x_1^7 x_3^3 x_4^9 x_5^{13}, & \widetilde{Z'_{122}} &= x_1^7 x_3^3 x_4^{13} x_5^9, & \widetilde{Z'_{123}} &= x_1^7 x_3^7 x_4^9 x_5^9, & \widetilde{Z'_{124}} &= x_1^7 x_3^{11} x_4 x_5^{13}, \\
\widetilde{Z'_{125}} &= x_1^7 x_3^{11} x_4^5 x_5^9, & \widetilde{Z'_{126}} &= x_1^7 x_3^{11} x_4^{13} x_5, & \widetilde{Z'_{127}} &= x_1^7 x_2 x_4^{11} x_5^{13}, & \widetilde{Z'_{128}} &= x_1^7 x_2 x_3^{11} x_5^{13}, \\
\widetilde{Z'_{129}} &= x_1^7 x_2 x_3^{11} x_4^{13}, & \widetilde{Z'_{130}} &= x_1^7 x_2^3 x_4^9 x_5^{13}, & \widetilde{Z'_{131}} &= x_1^7 x_2^3 x_4^{13} x_5^9, & \widetilde{Z'_{132}} &= x_1^7 x_2^3 x_3^9 x_5^{13}, \\
\widetilde{Z'_{133}} &= x_1^7 x_2^3 x_3^9 x_4^{13}, & \widetilde{Z'_{134}} &= x_1^7 x_2^3 x_3^{13} x_5^9, & \widetilde{Z'_{135}} &= x_1^7 x_2^3 x_3^{13} x_4^9, & \widetilde{Z'_{136}} &= x_1^7 x_2^7 x_4^9 x_5^9, \\
\widetilde{Z'_{137}} &= x_1^7 x_2^7 x_3^9 x_5^9, & \widetilde{Z'_{138}} &= x_1^7 x_2^7 x_3^9 x_4^9, & \widetilde{Z'_{139}} &= x_1^7 x_2^{11} x_4 x_5^{13}, & \widetilde{Z'_{140}} &= x_1^7 x_2^{11} x_4^5 x_5^9, \\
\widetilde{Z'_{141}} &= x_1^7 x_2^{11} x_4^{13} x_5, & \widetilde{Z'_{142}} &= x_1^7 x_2^{11} x_3 x_5^{13}, & \widetilde{Z'_{143}} &= x_1^7 x_2^{11} x_3 x_4^{13}, & \widetilde{Z'_{144}} &= x_1^7 x_2^{11} x_3^5 x_5^9, \\
\widetilde{Z'_{145}} &= x_1^7 x_2^{11} x_3^9 x_4^9, & \widetilde{Z'_{146}} &= x_1^7 x_2^{11} x_3^{13} x_5, & \widetilde{Z'_{147}} &= x_1^7 x_2^{11} x_3^{13} x_4, & \widetilde{Z'_{148}} &= x_1^{15} x_3 x_4 x_5^{15}, \\
\widetilde{Z'_{149}} &= x_1^{15} x_3 x_4^3 x_5^{13}, & \widetilde{Z'_{150}} &= x_1^{15} x_3 x_4^{15} x_5, & \widetilde{Z'_{151}} &= x_1^{15} x_3^3 x_4 x_5^{13}, & \widetilde{Z'_{152}} &= x_1^{15} x_3^3 x_4^5 x_5^9, \\
\widetilde{Z'_{153}} &= x_1^{15} x_3^3 x_4^{13} x_5, & \widetilde{Z'_{154}} &= x_1^{15} x_3^{15} x_4 x_5, & \widetilde{Z'_{155}} &= x_1^{15} x_2 x_4 x_5^{15}, & \widetilde{Z'_{156}} &= x_1^{15} x_2 x_3^3 x_5^{13}, \\
\widetilde{Z'_{157}} &= x_1^{15} x_2 x_4^{15} x_5, & \widetilde{Z'_{158}} &= x_1^{15} x_2 x_3 x_5^{15}, & \widetilde{Z'_{159}} &= x_1^{15} x_2 x_3 x_4^{15}, & \widetilde{Z'_{160}} &= x_1^{15} x_2 x_3^3 x_5^{13}, \\
\widetilde{Z'_{161}} &= x_1^{15} x_2 x_3^3 x_4^{13}, & \widetilde{Z'_{162}} &= x_1^{15} x_2 x_3^{15} x_5, & \widetilde{Z'_{163}} &= x_1^{15} x_2 x_3^{15} x_4, & \widetilde{Z'_{164}} &= x_1^{15} x_2^3 x_4 x_5^{13}, \\
\widetilde{Z'_{165}} &= x_1^{15} x_2^3 x_4^5 x_5^9, & \widetilde{Z'_{166}} &= x_1^{15} x_2^3 x_4^{13} x_5, & \widetilde{Z'_{167}} &= x_1^{15} x_2^3 x_3 x_5^{13}, & \widetilde{Z'_{168}} &= x_1^{15} x_2^3 x_3 x_4^{13}, \\
\widetilde{Z'_{169}} &= x_1^{15} x_2^3 x_3^5 x_5^9, & \widetilde{Z'_{170}} &= x_1^{15} x_2^3 x_3^9 x_4^9, & \widetilde{Z'_{171}} &= x_1^{15} x_2^3 x_3^{13} x_5, & \widetilde{Z'_{172}} &= x_1^{15} x_2^3 x_3^{13} x_4, \\
\widetilde{Z'_{173}} &= x_1^{15} x_2^{15} x_4 x_5, & \widetilde{Z'_{174}} &= x_1^{15} x_2^{15} x_3 x_5, & \widetilde{Z'_{175}} &= x_1^{15} x_2^{15} x_3 x_4, & \widetilde{Z'_1} &= x_1 x_2 x_3 x_4^{14} x_5^{15}, \\
\widetilde{Z'_2} &= x_1 x_2 x_3 x_4^{15} x_5^{14}, & \widetilde{Z'_3} &= x_1 x_2 x_3^2 x_4^{13} x_5^{15}, & \widetilde{Z'_4} &= x_1 x_2 x_3^2 x_4^{15} x_5^{13}, & \widetilde{Z'_5} &= x_1 x_2 x_3^3 x_4^{12} x_5^{15}, \\
\widetilde{Z'_6} &= x_1 x_2 x_3^3 x_4^{13} x_5^{14}, & \widetilde{Z'_7} &= x_1 x_2 x_3^3 x_4^{14} x_5^{13}, & \widetilde{Z'_8} &= x_1 x_2 x_3^3 x_4^{15} x_5^{12}, & \widetilde{Z'_9} &= x_1 x_2 x_3^6 x_4^{11} x_5^{13}, \\
\widetilde{Z'_{10}} &= x_1 x_2 x_3^7 x_4^{10} x_5^{13}, & \widetilde{Z'_{11}} &= x_1 x_2 x_3^7 x_4^{11} x_5^{12}, & \widetilde{Z'_{12}} &= x_1 x_2 x_3^{14} x_4 x_5^{15}, & \widetilde{Z'_{13}} &= x_1 x_2 x_3^{14} x_4^3 x_5^{13}, \\
\widetilde{Z'_{14}} &= x_1 x_2 x_3^{14} x_4^{15} x_5, & \widetilde{Z'_{15}} &= x_1 x_2 x_3^{15} x_4 x_5^{14}, & \widetilde{Z'_{16}} &= x_1 x_2 x_3^{15} x_4^2 x_5^{13}, & \widetilde{Z'_{17}} &= x_1 x_2 x_3^{15} x_4^3 x_5^{12}, \\
\widetilde{Z'_{18}} &= x_1 x_2 x_3^{15} x_4^{14} x_5, & \widetilde{Z'_{19}} &= x_1 x_2^2 x_3 x_4^{13} x_5^{15}, & \widetilde{Z'_{20}} &= x_1 x_2^2 x_3 x_4^{15} x_5^{13}, & \widetilde{Z'_{21}} &= x_1 x_2^2 x_3^3 x_4^{13} x_5^{13}, \\
\widetilde{Z'_{22}} &= x_1 x_2^2 x_3^5 x_4^{15} x_5, & \widetilde{Z'_{23}} &= x_1 x_2^2 x_3^9 x_4^{11} x_5^{13}, & \widetilde{Z'_{24}} &= x_1 x_2^2 x_3^5 x_4^{15} x_5^9, & \widetilde{Z'_{25}} &= x_1 x_2^2 x_3^7 x_4^9 x_5^{13},
\end{aligned}$$

$$\begin{aligned}
Z'_{26} &= x_1 x_2^2 x_3^7 x_4^{13} x_5^9, & Z'_{27} &= x_1 x_2^2 x_3^{13} x_4 x_5^{15}, & Z'_{28} &= x_1 x_2^2 x_3^{13} x_4^3 x_5^{13}, & Z'_{29} &= x_1 x_2^2 x_3^{13} x_4^{15} x_5, \\
Z'_{30} &= x_1 x_2^2 x_3^{15} x_4 x_5^{13}, & Z'_{31} &= x_1 x_2^2 x_3^{15} x_4^5 x_5^9, & Z'_{32} &= x_1 x_2^2 x_3^{15} x_4^{13} x_5, & Z'_{33} &= x_1 x_2^3 x_3 x_4^{12} x_5^{15}, \\
Z'_{34} &= x_1 x_2^3 x_3 x_4^{13} x_5^{14}, & Z'_{35} &= x_1 x_2^3 x_3 x_4^{14} x_5^{13}, & Z'_{36} &= x_1 x_2^3 x_3 x_4^{15} x_5^{12}, & Z'_{37} &= x_1 x_2^3 x_3^2 x_4^{13} x_5^{13}, \\
Z'_{38} &= x_1 x_2^3 x_3^3 x_4^{12} x_5^{13}, & Z'_{39} &= x_1 x_2^3 x_3^3 x_4^{13} x_5^{12}, & Z'_{40} &= x_1 x_2^3 x_3^4 x_4^9 x_5^{15}, & Z'_{41} &= x_1 x_2^3 x_3^4 x_4^{11} x_5^{13}, \\
Z'_{42} &= x_1 x_2^3 x_3^4 x_4^{15} x_5^9, & Z'_{43} &= x_1 x_2^3 x_3^5 x_4^8 x_5^{15}, & Z'_{44} &= x_1 x_2^3 x_3^5 x_4^9 x_5^{14}, & Z'_{45} &= x_1 x_2^3 x_3^5 x_4^{10} x_5^{13}, \\
Z'_{46} &= x_1 x_2^3 x_3^5 x_4^{12} x_5^{12}, & Z'_{47} &= x_1 x_2^3 x_3^5 x_4^{14} x_5^9, & Z'_{48} &= x_1 x_2^3 x_3^5 x_4^{15} x_5^8, & Z'_{49} &= x_1 x_2^3 x_3^6 x_4^9 x_5^{13}, \\
Z'_{50} &= x_1 x_2^3 x_3^6 x_4^{13} x_5^9, & Z'_{51} &= x_1 x_2^3 x_3^7 x_4^8 x_5^{13}, & Z'_{52} &= x_1 x_2^3 x_3^7 x_4^9 x_5^{12}, & Z'_{53} &= x_1 x_2^3 x_3^7 x_4^{12} x_5^9, \\
Z'_{54} &= x_1 x_2^3 x_3^7 x_4^{13} x_5^8, & Z'_{55} &= x_1 x_2^3 x_3^{12} x_4 x_5^{15}, & Z'_{56} &= x_1 x_2^3 x_3^{12} x_4^3 x_5^{13}, & Z'_{57} &= x_1 x_2^3 x_3^{12} x_4^{15} x_5, \\
Z'_{58} &= x_1 x_2^3 x_3^{13} x_4 x_5^{14}, & Z'_{59} &= x_1 x_2^3 x_3^{13} x_4^2 x_5^{13}, & Z'_{60} &= x_1 x_2^3 x_3^{13} x_4^3 x_5^{12}, & Z'_{61} &= x_1 x_2^3 x_3^{13} x_4^{14} x_5, \\
Z'_{62} &= x_1 x_2^3 x_3^{14} x_4 x_5^{13}, & Z'_{63} &= x_1 x_2^3 x_3^{14} x_4^5 x_5^9, & Z'_{64} &= x_1 x_2^3 x_3^{14} x_4^{13} x_5, & Z'_{65} &= x_1 x_2^3 x_3^{15} x_4 x_5^{12}, \\
Z'_{66} &= x_1 x_2^3 x_3^{15} x_4^4 x_5^9, & Z'_{67} &= x_1 x_2^3 x_3^{15} x_4^5 x_5^8, & Z'_{68} &= x_1 x_2^3 x_3^{15} x_4^{12} x_5, & Z'_{69} &= x_1 x_2^6 x_3 x_4^{11} x_5^{13}, \\
Z'_{70} &= x_1 x_2^6 x_3^9 x_4^{13}, & Z'_{71} &= x_1 x_2^6 x_3^{13} x_4 x_5, & Z'_{72} &= x_1 x_2^6 x_3^9 x_4^9, & Z'_{73} &= x_1 x_2^6 x_3^{11} x_4 x_5^{13}, \\
Z'_{74} &= x_1 x_2^6 x_3^{11} x_4^5 x_5^9, & Z'_{75} &= x_1 x_2^6 x_3^{11} x_4^{13} x_5, & Z'_{76} &= x_1 x_2^7 x_3 x_4^{10} x_5^{13}, & Z'_{77} &= x_1 x_2^7 x_3 x_4^{11} x_5^{12}, \\
Z'_{78} &= x_1 x_2^7 x_3^9 x_4^{13}, & Z'_{79} &= x_1 x_2^7 x_3^{13} x_4^9 x_5, & Z'_{80} &= x_1 x_2^7 x_3^8 x_4^{13}, & Z'_{81} &= x_1 x_2^7 x_3^9 x_4^{12}, \\
Z'_{82} &= x_1 x_2^7 x_3^{12} x_4^9, & Z'_{83} &= x_1 x_2^7 x_3^{13} x_4^8 x_5, & Z'_{84} &= x_1 x_2^7 x_3^6 x_4^9 x_5^9, & Z'_{85} &= x_1 x_2^7 x_3^8 x_4^9 x_5, \\
Z'_{86} &= x_1 x_2^7 x_3^9 x_4^8 x_5, & Z'_{87} &= x_1 x_2^7 x_3^{10} x_4 x_5^{13}, & Z'_{88} &= x_1 x_2^7 x_3^{10} x_4^5 x_5^9, & Z'_{89} &= x_1 x_2^7 x_3^{10} x_4^{13} x_5, \\
Z'_{90} &= x_1 x_2^7 x_3^{11} x_4 x_5^{12}, & Z'_{91} &= x_1 x_2^7 x_3^{11} x_4^4 x_5^9, & Z'_{92} &= x_1 x_2^7 x_3^{11} x_4^5 x_5^8, & Z'_{93} &= x_1 x_2^7 x_3^{11} x_4^{12} x_5, \\
Z'_{94} &= x_1 x_2^{14} x_3 x_4 x_5^{15}, & Z'_{95} &= x_1 x_2^{14} x_3 x_4^3 x_5^{13}, & Z'_{96} &= x_1 x_2^{14} x_3 x_4^{15} x_5, & Z'_{97} &= x_1 x_2^{14} x_3^3 x_4 x_5^{13}, \\
Z'_{98} &= x_1 x_2^{14} x_3^5 x_4^9 x_5, & Z'_{99} &= x_1 x_2^{14} x_3^3 x_4^{13} x_5, & Z'_{100} &= x_1 x_2^{14} x_3^3 x_4 x_5^{15}, & Z'_{101} &= x_1 x_2^{15} x_3 x_4 x_5^{14}, \\
Z'_{102} &= x_1 x_2^{15} x_3 x_4^2 x_5^{13}, & Z'_{103} &= x_1 x_2^{15} x_3 x_4^3 x_5^{12}, & Z'_{104} &= x_1 x_2^{15} x_3 x_4^{14} x_5, & Z'_{105} &= x_1 x_2^{15} x_3^2 x_4 x_5^{13}, \\
Z'_{106} &= x_1 x_2^{15} x_3^2 x_4^5 x_5^9, & Z'_{107} &= x_1 x_2^{15} x_3^2 x_4^{13} x_5, & Z'_{108} &= x_1 x_2^{15} x_3^3 x_4 x_5^{12}, & Z'_{109} &= x_1 x_2^{15} x_3^3 x_4^9 x_5, \\
Z'_{110} &= x_1 x_2^{15} x_3^3 x_4^5 x_5^8, & Z'_{111} &= x_1 x_2^{15} x_3^3 x_4^{12} x_5, & Z'_{112} &= x_1 x_2^{15} x_3^{14} x_4 x_5, & Z'_{113} &= x_1^3 x_2 x_3 x_4^{12} x_5^{15}, \\
Z'_{114} &= x_1^3 x_2 x_3 x_4^{13} x_5^{14}, & Z'_{115} &= x_1^3 x_2 x_3 x_4^{14} x_5^{13}, & Z'_{116} &= x_1^3 x_2 x_3 x_4^{15} x_5^{12}, & Z'_{117} &= x_1^3 x_2 x_3^2 x_4^{13} x_5^{13}, \\
Z'_{118} &= x_1^3 x_2 x_3^3 x_4^{12} x_5^{13}, & Z'_{119} &= x_1^3 x_2 x_3^3 x_4^{13} x_5^{12}, & Z'_{120} &= x_1^3 x_2 x_3^4 x_4^9 x_5^{15}, & Z'_{121} &= x_1^3 x_2 x_3^4 x_4^{11} x_5^{13}, \\
Z'_{122} &= x_1^3 x_2 x_3^4 x_4^{15} x_5^9, & Z'_{123} &= x_1^3 x_2 x_3^5 x_4^8 x_5^{15}, & Z'_{124} &= x_1^3 x_2 x_3^5 x_4^9 x_5^{14}, & Z'_{125} &= x_1^3 x_2 x_3^5 x_4^{10} x_5^{13}, \\
Z'_{126} &= x_1^3 x_2 x_3^5 x_4^{11} x_5^{12}, & Z'_{127} &= x_1^3 x_2 x_3^5 x_4^{14} x_5^9, & Z'_{128} &= x_1^3 x_2 x_3^5 x_4^{15} x_5^8, & Z'_{129} &= x_1^3 x_2 x_3^6 x_4^9 x_5^{13}, \\
Z'_{130} &= x_1^3 x_2 x_3^6 x_4^{13} x_5^9, & Z'_{131} &= x_1^3 x_2 x_3^7 x_4^8 x_5^{15}, & Z'_{132} &= x_1^3 x_2 x_3^7 x_4^9 x_5^{12}, & Z'_{133} &= x_1^3 x_2 x_3^7 x_4^{12} x_5^9, \\
Z'_{134} &= x_1^3 x_2 x_3^7 x_4^{13} x_5^8, & Z'_{135} &= x_1^3 x_2 x_3^{12} x_4 x_5^{15}, & Z'_{136} &= x_1^3 x_2 x_3^{12} x_4^3 x_5^{13}, & Z'_{137} &= x_1^3 x_2 x_3^{12} x_4^{15} x_5, \\
Z'_{138} &= x_1^3 x_2 x_3^{13} x_4 x_5^{14}, & Z'_{139} &= x_1^3 x_2 x_3^{13} x_4^2 x_5^{13}, & Z'_{140} &= x_1^3 x_2 x_3^{13} x_4^3 x_5^{12}, & Z'_{141} &= x_1^3 x_2 x_3^{13} x_4^{14} x_5, \\
Z'_{142} &= x_1^3 x_2 x_3^{14} x_4 x_5^{13}, & Z'_{143} &= x_1^3 x_2 x_3^{14} x_4^5 x_5^9, & Z'_{144} &= x_1^3 x_2 x_3^{14} x_4^{13} x_5, & Z'_{145} &= x_1^3 x_2 x_3^{15} x_4 x_5^{12}, \\
Z'_{146} &= x_1^3 x_2 x_3^{15} x_4^4 x_5^9, & Z'_{147} &= x_1^3 x_2 x_3^{15} x_4^5 x_5^8, & Z'_{148} &= x_1^3 x_2 x_3^{15} x_4^{12} x_5, & Z'_{149} &= x_1^3 x_2^3 x_3 x_4^{12} x_5^{13}, \\
Z'_{150} &= x_1^3 x_2^3 x_3 x_4^{13} x_5^{12}, & Z'_{151} &= x_1^3 x_2^3 x_3^4 x_4^9 x_5^{13}, & Z'_{152} &= x_1^3 x_2^3 x_3^4 x_4^{13} x_5^9, & Z'_{153} &= x_1^3 x_2^3 x_3^5 x_4^8 x_5^{13}, \\
Z'_{154} &= x_1^3 x_2^3 x_3^5 x_4^9 x_5^{12}, & Z'_{155} &= x_1^3 x_2^3 x_3^5 x_4^{12} x_5^9, & Z'_{156} &= x_1^3 x_2^3 x_3^5 x_4^{13} x_5^8, & Z'_{157} &= x_1^3 x_2^3 x_3^{12} x_4 x_5^{13}, \\
Z'_{158} &= x_1^3 x_2^3 x_3^{12} x_4^5 x_5^9, & Z'_{159} &= x_1^3 x_2^3 x_3^{12} x_4^{13} x_5, & Z'_{160} &= x_1^3 x_2^3 x_3^{13} x_4 x_5^{12}, & Z'_{161} &= x_1^3 x_2^3 x_3^{13} x_4^9 x_5, \\
Z'_{162} &= x_1^3 x_2^3 x_3^{13} x_4^5 x_5^8, & Z'_{163} &= x_1^3 x_2^3 x_3^{13} x_4^{12} x_5, & Z'_{164} &= x_1^3 x_2^5 x_3 x_4^8 x_5^{15}, & Z'_{165} &= x_1^3 x_2^5 x_3 x_4^9 x_5^{14}, \\
Z'_{166} &= x_1^3 x_2^5 x_3 x_4^{10} x_5^{13}, & Z'_{167} &= x_1^3 x_2^5 x_3 x_4^{11} x_5^{12}, & Z'_{168} &= x_1^3 x_2^5 x_3 x_4^{14} x_5^9, & Z'_{169} &= x_1^3 x_2^5 x_3 x_4^{15} x_5^8, \\
Z'_{170} &= x_1^3 x_2^5 x_3^2 x_4^9 x_5^{13}, & Z'_{171} &= x_1^3 x_2^5 x_3^2 x_4^{13} x_5^9, & Z'_{172} &= x_1^3 x_2^5 x_3^3 x_4^8 x_5^{13}, & Z'_{173} &= x_1^3 x_2^5 x_3^3 x_4^9 x_5^{12}, \\
Z'_{174} &= x_1^3 x_2^5 x_3^3 x_4^{12} x_5^9, & Z'_{175} &= x_1^3 x_2^5 x_3^3 x_4^{13} x_5^8, & Z'_{176} &= x_1^3 x_2^5 x_3^6 x_4^9 x_5^9, & Z'_{177} &= x_1^3 x_2^5 x_3^7 x_4^8 x_5^9, \\
Z'_{178} &= x_1^3 x_2^5 x_3^7 x_4^9 x_5^8, & Z'_{179} &= x_1^3 x_2^5 x_3^9 x_4 x_5^{14}, & Z'_{180} &= x_1^3 x_2^5 x_3^{10} x_4 x_5^{13}, & Z'_{181} &= x_1^3 x_2^5 x_3^{10} x_4^5 x_5^9, \\
Z'_{182} &= x_1^3 x_2^5 x_3^{10} x_4 x_5^{13}, & Z'_{183} &= x_1^3 x_2^5 x_3^{11} x_4 x_5^{12}, & Z'_{184} &= x_1^3 x_2^5 x_3^{11} x_4^4 x_5^9, & Z'_{185} &= x_1^3 x_2^5 x_3^{11} x_4^5 x_5^8, \\
Z'_{186} &= x_1^3 x_2^5 x_3^{11} x_4^5 x_5^{12}, & Z'_{187} &= x_1^3 x_2^5 x_3^{15} x_4 x_5^8, & Z'_{188} &= x_1^3 x_2^7 x_3 x_4^8 x_5^{13}, & Z'_{189} &= x_1^3 x_2^7 x_3 x_4^9 x_5^{12}, \\
Z'_{190} &= x_1^3 x_2^7 x_3 x_4^{12} x_5^9, & Z'_{191} &= x_1^3 x_2^7 x_3 x_4^{13} x_5^8, & Z'_{192} &= x_1^3 x_2^7 x_3^5 x_4^8 x_5^9, & Z'_{193} &= x_1^3 x_2^7 x_3^5 x_4^9 x_5^8, \\
Z'_{194} &= x_1^3 x_2^7 x_3^9 x_4^{12} x_5^5, & Z'_{195} &= x_1^3 x_2^7 x_3^9 x_4^4 x_5^9, & Z'_{196} &= x_1^3 x_2^7 x_3^9 x_4^5 x_5^8, & Z'_{197} &= x_1^3 x_2^7 x_3^9 x_4^{12} x_5^5, \\
Z'_{198} &= x_1^3 x_2^7 x_3^{13} x_4 x_5^8, & Z'_{199} &= x_1^3 x_2^{13} x_3 x_4 x_5^{14}, & Z'_{200} &= x_1^3 x_2^{13} x_3 x_4^2 x_5^{13}, & Z'_{201} &= x_1^3 x_2^{13} x_3 x_4^3 x_5^{12}, \\
Z'_{202} &= x_1^3 x_2^{13} x_3 x_4^{14} x_5, & Z'_{203} &= x_1^3 x_2^{13} x_3^2 x_4 x_5^{13}, & Z'_{204} &= x_1^3 x_2^{13} x_3^2 x_4^5 x_5^9, & Z'_{205} &= x_1^3 x_2^{13} x_3^2 x_4^{13} x_5, \\
Z'_{206} &= x_1^3 x_2^{13} x_3^3 x_4 x_5^{12}, & Z'_{207} &= x_1^3 x_2^{13} x_3^3 x_4^9 x_5^9, & Z'_{208} &= x_1^3 x_2^{13} x_3^3 x_4^{15} x_5^8, & Z'_{209} &= x_1^3 x_2^{13} x_3^3 x_4^{12} x_5^8, \\
Z'_{210} &= x_1^3 x_2^{13} x_3^{14} x_4 x_5, & Z'_{211} &= x_1^3 x_2^{15} x_3 x_4 x_5^{12}, & Z'_{212} &= x_1^3 x_2^{15} x_3 x_4^8 x_5^9, & Z'_{213} &= x_1^3 x_2^{15} x_3 x_4^{11} x_5^{12}, \\
Z'_{214} &= x_1^3 x_2^{15} x_3 x_4^{12} x_5^8, & Z'_{215} &= x_1^3 x_2^{15} x_3^5 x_4 x_5^8, & Z'_{216} &= x_1^7 x_2 x_3 x_4 x_5^{10} x_5^{13}, & Z'_{217} &= x_1^7 x_2 x_3 x_4 x_5^{11} x_5^{12},
\end{aligned}$$

$$\begin{aligned}
Z'_{218} &= x_1^7 x_2 x_3^2 x_4^9 x_5^{13}, & Z'_{219} &= x_1^7 x_2 x_3^2 x_4^{13} x_5^9, & Z'_{220} &= x_1^7 x_2 x_3^3 x_4^8 x_5^{13}, & Z'_{221} &= x_1^7 x_2 x_3^3 x_4^9 x_5^{12}, \\
Z'_{222} &= x_1^7 x_2 x_3^3 x_4^{12} x_5^9, & Z'_{223} &= x_1^7 x_2 x_3^3 x_4^{13} x_5^8, & Z'_{224} &= x_1^7 x_2 x_3^6 x_4^9 x_5^9, & Z'_{225} &= x_1^7 x_2 x_3^7 x_4^8 x_5^9, \\
Z'_{226} &= x_1^7 x_2 x_3^7 x_4^9 x_5^8, & Z'_{227} &= x_1^7 x_2 x_3^{10} x_4^{13} x_5^9, & Z'_{228} &= x_1^7 x_2 x_3^{10} x_4^5 x_5^9, & Z'_{229} &= x_1^7 x_2 x_3^{10} x_4^{13} x_5^9, \\
Z'_{230} &= x_1^7 x_2 x_3^{11} x_4^5 x_5^{12}, & Z'_{231} &= x_1^7 x_2 x_3^{11} x_4^9 x_5^9, & Z'_{232} &= x_1^7 x_2 x_3^{11} x_4^5 x_5^8, & Z'_{233} &= x_1^7 x_2 x_3^{11} x_4^{12} x_5^9, \\
Z'_{234} &= x_1^7 x_2^3 x_3^8 x_4^{13} x_5^9, & Z'_{235} &= x_1^7 x_2^3 x_3^9 x_4^5 x_5^{12}, & Z'_{236} &= x_1^7 x_2^3 x_3^9 x_4^{12} x_5^9, & Z'_{237} &= x_1^7 x_2^3 x_3^9 x_4^{13} x_5^8, \\
Z'_{238} &= x_1^7 x_2^3 x_3^9 x_4^5 x_5^9, & Z'_{239} &= x_1^7 x_2^3 x_3^9 x_4^8 x_5^9, & Z'_{240} &= x_1^7 x_2^3 x_3^9 x_4^5 x_5^{12}, & Z'_{241} &= x_1^7 x_2^3 x_3^9 x_4^5 x_5^9, \\
Z'_{242} &= x_1^7 x_2^3 x_3^9 x_4^5 x_5^8, & Z'_{243} &= x_1^7 x_2^3 x_3^9 x_4^{12} x_5^9, & Z'_{244} &= x_1^7 x_2^3 x_3^{13} x_4^5 x_5^8, & Z'_{245} &= x_1^7 x_2^3 x_3^8 x_4^5 x_5^9, \\
Z'_{246} &= x_1^7 x_2^7 x_3^9 x_4^5 x_5^8, & Z'_{247} &= x_1^7 x_2^3 x_3^9 x_4^5 x_5^8, & Z'_{248} &= x_1^7 x_2^{11} x_3 x_4^5 x_5^{12}, & Z'_{249} &= x_1^7 x_2^{11} x_3 x_4^5 x_5^9, \\
Z'_{250} &= x_1^7 x_2^{11} x_3 x_4^5 x_5^8, & Z'_{251} &= x_1^7 x_2^{11} x_3 x_4^{12} x_5^9, & Z'_{252} &= x_1^7 x_2^{11} x_3^5 x_4^5 x_5^8, & Z'_{253} &= x_1^{15} x_2 x_3 x_4^5 x_5^{14}, \\
Z'_{254} &= x_1^{15} x_2 x_3 x_4^2 x_5^{13}, & Z'_{255} &= x_1^{15} x_2 x_3 x_4^3 x_5^{12}, & Z'_{256} &= x_1^{15} x_2 x_3 x_4^{14} x_5^9, & Z'_{257} &= x_1^{15} x_2 x_3^2 x_4^5 x_5^{13}, \\
Z'_{258} &= x_1^{15} x_2 x_3^2 x_4^5 x_5^9, & Z'_{259} &= x_1^{15} x_2 x_3^2 x_4^{13} x_5^9, & Z'_{260} &= x_1^{15} x_2 x_3^3 x_4^5 x_5^{12}, & Z'_{261} &= x_1^{15} x_2 x_3^3 x_4^5 x_5^9, \\
Z'_{262} &= x_1^{15} x_2 x_3^3 x_4^5 x_5^8, & Z'_{263} &= x_1^{15} x_2 x_3^3 x_4^{12} x_5^9, & Z'_{264} &= x_1^{15} x_2 x_3^{14} x_4^5 x_5^8, & Z'_{265} &= x_1^{15} x_2 x_3^3 x_4^5 x_5^{12}, \\
Z'_{266} &= x_1^{15} x_2^3 x_3 x_4^5 x_5^9, & Z'_{267} &= x_1^{15} x_2^3 x_3 x_4^5 x_5^8, & Z'_{268} &= x_1^{15} x_2^3 x_3 x_4^{12} x_5^9, & Z'_{269} &= x_1^{15} x_2^3 x_3^5 x_4^5 x_5^8, \\
Z'_{270} &= x_1^3 x_2^{15} x_3^5 x_4^5 x_5^9, & Z'_{271} &= x_1^3 x_2^4 x_3 x_4^{11} x_5^{13}, & Z'_{272} &= x_1^3 x_2^5 x_3^{15} x_4^5 x_5^8, & Z'_{273} &= x_1^3 x_2^4 x_3^{11} x_4^{13} x_5^9, \\
Z'_{274} &= x_1^3 x_2^7 x_3^8 x_4^{13} x_5^9, & Z'_{275} &= x_1^3 x_2^5 x_3^8 x_4^5 x_5^{15}, & Z'_{276} &= x_1^3 x_2^4 x_3^{11} x_4^5 x_5^{13}, & Z'_{277} &= x_1^3 x_2^7 x_3^8 x_4^5 x_5^{13}, \\
Z'_{278} &= x_1^3 x_2^5 x_3^8 x_4^{15} x_5^9, & Z'_{279} &= x_1^3 x_2^3 x_3^8 x_4^{13} x_5^9, & Z'_{280} &= x_1^3 x_2^7 x_3^8 x_4^5 x_5^9, & Z'_{281} &= x_1^3 x_2^7 x_3^9 x_4^5 x_5^8, \\
Z'_{282} &= x_1^3 x_2^3 x_3^8 x_4^5 x_5^{13}, & Z'_{283} &= x_1^3 x_2^7 x_3^8 x_4^5 x_5^9, & Z'_{284} &= x_1^{15} x_2^3 x_3^5 x_4^5 x_5^8, & Z'_{285} &= x_1^3 x_2^4 x_3^3 x_4^5 x_5^{13}, \\
Z'_{286} &= x_1^3 x_2^4 x_3^3 x_4^{13} x_5^9, & Z'_{287} &= x_1^3 x_2^7 x_3^9 x_4^5 x_5^9, & Z'_{288} &= x_1^3 x_2^4 x_3^{11} x_4^5 x_5^9, & Z'_{289} &= x_1^3 x_2^5 x_3^8 x_4^5 x_5^{13}, \\
Z'_{290} &= x_1^3 x_2^5 x_3^9 x_4^5 x_5^{13}, & Z'_{291} &= x_1^3 x_2^5 x_3^9 x_4^5 x_5^{12}, & Z'_{292} &= x_1^3 x_2^5 x_3^9 x_4^5 x_5^9, & Z'_{293} &= x_1^3 x_2^5 x_3^9 x_4^5 x_5^8, \\
Z'_{294} &= x_1^3 x_2^5 x_3^9 x_4^5 x_5^{14}, & Z'_{295} &= x_1^3 x_2^7 x_3^9 x_4^5 x_5^9, & Z'_{296} &= x_1^3 x_2^7 x_3^8 x_4^5 x_5^9, & Z'_{297} &= x_1^3 x_2^7 x_3^{12} x_4^5 x_5^9, \\
Z'_{298} &= x_1^3 x_2^7 x_3^{12} x_4^5 x_5^9, & Z'_{299} &= x_1^3 x_2^7 x_3^{13} x_4^5 x_5^9, & Z'_{300} &= x_1^3 x_2^{12} x_3^3 x_4^5 x_5^9, & Z'_{301} &= x_1^7 x_2^3 x_4^5 x_5^9, \\
Z'_{302} &= x_1^7 x_2^3 x_3^8 x_4^5 x_5^9, & Z'_{303} &= x_1^7 x_2^3 x_3^{12} x_4^5 x_5^9, & Z'_{304} &= x_1^7 x_2^3 x_3^{12} x_4^5 x_5^9, & Z'_{305} &= x_1^7 x_2^3 x_3^{13} x_4^5 x_5^8, \\
Z'_{306} &= x_1^7 x_2^8 x_3^3 x_4^5 x_5^9, & Z'_{307} &= x_1^7 x_2^9 x_3^2 x_4^5 x_5^9, & Z'_{308} &= x_1^7 x_2^9 x_3^3 x_4^5 x_5^9, & Z'_{309} &= x_1^7 x_2^9 x_3^3 x_4^5 x_5^8, \\
Z'_{310} &= x_1^7 x_2^{11} x_3^5 x_4^5 x_5^8.
\end{aligned}$$

3.8. Admissible monomials in $(\mathcal{C}_{32}^{\otimes 5})^{\omega'_{(3)}}$

We have $(\mathcal{C}_{32}^{\otimes 5})^{\omega'_{(3)}} = (\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(3)})^0} \cup (\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(3)})^{>0}}$, where

$$\begin{aligned}
(\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(3)})^0} &= A(\omega'_{(3)}) = \{\widetilde{Z}_j'' : 1 \leq j \leq 75\}, \\
(\mathcal{C}_{32}^{\otimes 5})^{(\omega'_{(3)})^{>0}} &= B(\omega'_{(3)}) \cup C(\omega'_{(3)}) = \{Z_j'' : 1 \leq j \leq 320\}.
\end{aligned}$$

The admissible monomials $\widetilde{Z}_j''$ and Z_j'' are determined as follows:

$$\begin{aligned}
\widetilde{Z}_1'' &= x_2^3 x_3^7 x_4^5 x_5^{15}, & \widetilde{Z}_2'' &= x_2^3 x_3^7 x_4^{15} x_5^7, & \widetilde{Z}_3'' &= x_2^3 x_3^{15} x_4^7 x_5^7, & \widetilde{Z}_4'' &= x_2^7 x_3^3 x_4^7 x_5^{15}, \\
\widetilde{Z}_5'' &= x_2^7 x_3^3 x_4^{15} x_5^7, & \widetilde{Z}_6'' &= x_2^7 x_3^7 x_4^5 x_5^{15}, & \widetilde{Z}_7'' &= x_2^7 x_3^7 x_4^5 x_5^{11}, & \widetilde{Z}_8'' &= x_2^7 x_3^7 x_4^{11} x_5^7, \\
\widetilde{Z}_9'' &= x_2^7 x_3^7 x_4^{15} x_5^3, & \widetilde{Z}_{10}'' &= x_2^7 x_3^{11} x_4^7 x_5^7, & \widetilde{Z}_{11}'' &= x_2^7 x_3^{15} x_4^3 x_5^7, & \widetilde{Z}_{12}'' &= x_2^7 x_3^{15} x_4^7 x_5^3, \\
\widetilde{Z}_{13}'' &= x_2^{15} x_3^3 x_4^7 x_5^7, & \widetilde{Z}_{14}'' &= x_2^{15} x_3^7 x_4^3 x_5^7, & \widetilde{Z}_{15}'' &= x_2^{15} x_3^7 x_4^5 x_5^3, & \widetilde{Z}_{16}'' &= x_1^3 x_3^7 x_4^5 x_5^{15}, \\
\widetilde{Z}_{17}'' &= x_1^3 x_3^7 x_4^{15} x_5^7, & \widetilde{Z}_{18}'' &= x_1^3 x_3^{15} x_4^7 x_5^7, & \widetilde{Z}_{19}'' &= x_1^3 x_3^7 x_4^7 x_5^{15}, & \widetilde{Z}_{20}'' &= x_1^3 x_3^7 x_4^{15} x_5^7, \\
\widetilde{Z}_{21}'' &= x_1^3 x_3^7 x_4^{15} x_5^3, & \widetilde{Z}_{22}'' &= x_1^3 x_3^7 x_4^{15} x_5^7, & \widetilde{Z}_{23}'' &= x_1^3 x_3^7 x_4^{15} x_5^7, & \widetilde{Z}_{24}'' &= x_1^3 x_3^7 x_4^{15} x_5^7, \\
\widetilde{Z}_{25}'' &= x_1^3 x_3^{15} x_4^7 x_5^7, & \widetilde{Z}_{26}'' &= x_1^3 x_3^{15} x_4^7 x_5^7, & \widetilde{Z}_{27}'' &= x_1^3 x_3^{15} x_4^7 x_5^7, & \widetilde{Z}_{28}'' &= x_1^7 x_3^3 x_4^7 x_5^{15}, \\
\widetilde{Z}_{29}'' &= x_1^7 x_3^3 x_4^{15} x_5^7, & \widetilde{Z}_{30}'' &= x_1^7 x_3^7 x_4^3 x_5^{15}, & \widetilde{Z}_{31}'' &= x_1^7 x_3^7 x_4^5 x_5^{11}, & \widetilde{Z}_{32}'' &= x_1^7 x_3^7 x_4^{11} x_5^7, \\
\widetilde{Z}_{33}'' &= x_1^7 x_3^7 x_4^{15} x_5^3, & \widetilde{Z}_{34}'' &= x_1^7 x_3^{11} x_4^7 x_5^7, & \widetilde{Z}_{35}'' &= x_1^7 x_3^{15} x_4^3 x_5^7, & \widetilde{Z}_{36}'' &= x_1^7 x_3^{15} x_4^7 x_5^3, \\
\widetilde{Z}_{37}'' &= x_1^7 x_3^3 x_4^7 x_5^{15}, & \widetilde{Z}_{38}'' &= x_1^7 x_3^3 x_4^{15} x_5^7, & \widetilde{Z}_{39}'' &= x_1^7 x_3^3 x_4^7 x_5^{15}, & \widetilde{Z}_{40}'' &= x_1^7 x_3^3 x_4^7 x_5^{15}, \\
\widetilde{Z}_{41}'' &= x_1^7 x_3^3 x_4^{15} x_5^7, & \widetilde{Z}_{42}'' &= x_1^7 x_3^3 x_4^{15} x_5^7, & \widetilde{Z}_{43}'' &= x_1^7 x_3^3 x_4^{15} x_5^7, & \widetilde{Z}_{44}'' &= x_1^7 x_3^3 x_4^{15} x_5^{11}, \\
\widetilde{Z}_{45}'' &= x_1^7 x_3^7 x_4^{11} x_5^7, & \widetilde{Z}_{46}'' &= x_1^7 x_3^7 x_4^{15} x_5^3, & \widetilde{Z}_{47}'' &= x_1^7 x_3^7 x_4^{15} x_5^3, & \widetilde{Z}_{48}'' &= x_1^7 x_3^7 x_4^{15} x_5^3,
\end{aligned}$$

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$$\begin{aligned}
Z''_{158} &= x_1^3 x_2^{15} x_3 x_4^6 x_5^7, & Z''_{159} &= x_1^3 x_2^{15} x_3 x_4^7 x_5^6, & Z''_{160} &= x_1^3 x_2^{15} x_3^3 x_4^4 x_5^7, & Z''_{161} &= x_1^3 x_2^{15} x_3^3 x_4^5 x_5^6, \\
Z''_{162} &= x_1^3 x_2^{15} x_3^3 x_4^7 x_5^4, & Z''_{163} &= x_1^3 x_2^{15} x_3^5 x_4^2 x_5^7, & Z''_{164} &= x_1^3 x_2^{15} x_3^5 x_4^3 x_5^6, & Z''_{165} &= x_1^3 x_2^{15} x_3^5 x_4^6 x_5^3, \\
Z''_{166} &= x_1^3 x_2^{15} x_3^5 x_4^7 x_5^2, & Z''_{167} &= x_1^3 x_2^{15} x_3^7 x_4^2 x_5^6, & Z''_{168} &= x_1^3 x_2^{15} x_3^7 x_4^3 x_5^4, & Z''_{169} &= x_1^3 x_2^{15} x_3^7 x_4^5 x_5^2, \\
Z''_{170} &= x_1^7 x_2^2 x_3^2 x_4^7 x_5^{15}, & Z''_{171} &= x_1^7 x_2^2 x_3^2 x_4^{15} x_5^7, & Z''_{172} &= x_1^7 x_2^2 x_3^3 x_4^6 x_5^{15}, & Z''_{173} &= x_1^7 x_2^2 x_3^3 x_4^7 x_5^{14}, \\
Z''_{174} &= x_1^7 x_2^2 x_3^3 x_4^4 x_5^7, & Z''_{175} &= x_1^7 x_2^2 x_3^3 x_4^{15} x_5^6, & Z''_{176} &= x_1^7 x_2^2 x_3^6 x_4^3 x_5^{15}, & Z''_{177} &= x_1^7 x_2^2 x_3^6 x_4^4 x_5^{11}, \\
Z''_{178} &= x_1^7 x_2^2 x_3^6 x_4^4 x_5^7, & Z''_{179} &= x_1^7 x_2^2 x_3^6 x_4^{15} x_5^3, & Z''_{180} &= x_1^7 x_2^2 x_3^7 x_4^2 x_5^{15}, & Z''_{181} &= x_1^7 x_2^2 x_3^7 x_4^3 x_5^{14}, \\
Z''_{182} &= x_1^7 x_2^2 x_3^6 x_4^{11} x_5^7, & Z''_{183} &= x_1^7 x_2^2 x_3^7 x_4^7 x_5^{10}, & Z''_{184} &= x_1^7 x_2^2 x_3^7 x_4^{10} x_5^7, & Z''_{185} &= x_1^7 x_2^2 x_3^7 x_4^{11} x_5^6, \\
Z''_{186} &= x_1^7 x_2^2 x_3^7 x_4^{14} x_5^3, & Z''_{187} &= x_1^7 x_2^2 x_3^7 x_4^{15} x_5^2, & Z''_{188} &= x_1^7 x_2^2 x_3^{10} x_4^7 x_5^7, & Z''_{189} &= x_1^7 x_2^2 x_3^{11} x_4^6 x_5^7, \\
Z''_{190} &= x_1^7 x_2^2 x_3^{11} x_4^7 x_5^6, & Z''_{191} &= x_1^7 x_2^2 x_3^{14} x_4^3 x_5^7, & Z''_{192} &= x_1^7 x_2^2 x_3^{14} x_4^4 x_5^3, & Z''_{193} &= x_1^7 x_2^2 x_3^{15} x_4^2 x_5^7, \\
Z''_{194} &= x_1^7 x_2^2 x_3^{15} x_4^3 x_5^6, & Z''_{195} &= x_1^7 x_2^2 x_3^{15} x_4^6 x_5^3, & Z''_{196} &= x_1^7 x_2^2 x_3^{15} x_4^7 x_5^2, & Z''_{197} &= x_1^7 x_2^2 x_3^{15} x_4^6 x_5^{15}, \\
Z''_{198} &= x_1^7 x_2^2 x_3^3 x_4^7 x_5^{14}, & Z''_{199} &= x_1^7 x_2^2 x_3^3 x_4^{14} x_5^7, & Z''_{200} &= x_1^7 x_2^2 x_3^3 x_4^{15} x_5^6, & Z''_{201} &= x_1^7 x_2^2 x_3^3 x_4^4 x_5^{15}, \\
Z''_{202} &= x_1^7 x_2^2 x_3^3 x_4^5 x_5^{14}, & Z''_{203} &= x_1^7 x_2^2 x_3^3 x_4^7 x_5^{12}, & Z''_{204} &= x_1^7 x_2^2 x_3^3 x_4^{12} x_5^7, & Z''_{205} &= x_1^7 x_2^2 x_3^3 x_4^{13} x_5^6, \\
Z''_{206} &= x_1^7 x_2^2 x_3^3 x_4^{14} x_5^4, & Z''_{207} &= x_1^7 x_2^2 x_3^3 x_4^{15} x_5^{15}, & Z''_{208} &= x_1^7 x_2^2 x_3^5 x_4^3 x_5^{14}, & Z''_{209} &= x_1^7 x_2^2 x_3^5 x_4^4 x_5^{11}, \\
Z''_{210} &= x_1^7 x_2^2 x_3^5 x_4^7 x_5^{10}, & Z''_{211} &= x_1^7 x_2^2 x_3^5 x_4^{10} x_5^7, & Z''_{212} &= x_1^7 x_2^2 x_3^5 x_4^{11} x_5^6, & Z''_{213} &= x_1^7 x_2^2 x_3^5 x_4^{14} x_5^3, \\
Z''_{214} &= x_1^7 x_2^2 x_3^5 x_4^{15} x_5^2, & Z''_{215} &= x_1^7 x_2^2 x_3^7 x_4^4 x_5^{14}, & Z''_{216} &= x_1^7 x_2^2 x_3^7 x_4^5 x_5^{12}, & Z''_{217} &= x_1^7 x_2^2 x_3^7 x_4^5 x_5^{10}, \\
Z''_{218} &= x_1^7 x_2^2 x_3^7 x_4^8 x_5^8, & Z''_{219} &= x_1^7 x_2^2 x_3^7 x_4^9 x_5^6, & Z''_{220} &= x_1^7 x_2^2 x_3^7 x_4^{13} x_5^2, & Z''_{221} &= x_1^7 x_2^2 x_3^9 x_4^6 x_5^7, \\
Z''_{222} &= x_1^7 x_2^2 x_3^9 x_4^7 x_5^6, & Z''_{223} &= x_1^7 x_2^2 x_3^{11} x_4^5 x_5^6, & Z''_{224} &= x_1^7 x_2^2 x_3^{13} x_4^2 x_5^7, & Z''_{225} &= x_1^7 x_2^2 x_3^{13} x_4^3 x_5^6, \\
Z''_{226} &= x_1^7 x_2^2 x_3^{13} x_4^6 x_5^3, & Z''_{227} &= x_1^7 x_2^2 x_3^{13} x_4^7 x_5^2, & Z''_{228} &= x_1^7 x_2^2 x_3^{15} x_4^6 x_5^6, & Z''_{229} &= x_1^7 x_2^2 x_3^{15} x_4^3 x_5^4, \\
Z''_{230} &= x_1^7 x_2^2 x_3^{15} x_4^2 x_5^2, & Z''_{231} &= x_1^7 x_2^2 x_3^{15} x_4^2 x_5^{15}, & Z''_{232} &= x_1^7 x_2^2 x_3^{15} x_4^3 x_5^{14}, & Z''_{233} &= x_1^7 x_2^2 x_3^{15} x_4^4 x_5^{11}, \\
Z''_{234} &= x_1^7 x_2^2 x_3^7 x_4^4 x_5^{10}, & Z''_{235} &= x_1^7 x_2^2 x_3^{10} x_4^7 x_5^7, & Z''_{236} &= x_1^7 x_2^2 x_3^{11} x_4^6 x_5^6, & Z''_{237} &= x_1^7 x_2^2 x_3^{11} x_4^4 x_5^3, \\
Z''_{238} &= x_1^7 x_2^2 x_3^{15} x_4^2 x_5^2, & Z''_{239} &= x_1^7 x_2^2 x_3^{14} x_4^5 x_5^{14}, & Z''_{240} &= x_1^7 x_2^2 x_3^{14} x_4^5 x_5^{12}, & Z''_{241} &= x_1^7 x_2^2 x_3^{15} x_4^5 x_5^{10}, \\
Z''_{242} &= x_1^7 x_2^2 x_3^7 x_4^8 x_5^8, & Z''_{243} &= x_1^7 x_2^2 x_3^9 x_4^6 x_5^6, & Z''_{244} &= x_1^7 x_2^2 x_3^{13} x_4^2 x_5^2, & Z''_{245} &= x_1^7 x_2^2 x_3^{13} x_4^3 x_5^{10}, \\
Z''_{246} &= x_1^7 x_2^2 x_3^8 x_4^8 x_5^8, & Z''_{247} &= x_1^7 x_2^2 x_3^8 x_4^3 x_5^3, & Z''_{248} &= x_1^7 x_2^2 x_3^9 x_4^2 x_5^2, & Z''_{249} &= x_1^7 x_2^2 x_3^9 x_4^2 x_5^7, \\
Z''_{250} &= x_1^7 x_2^2 x_3^9 x_4^6 x_5^6, & Z''_{251} &= x_1^7 x_2^2 x_3^9 x_4^6 x_5^3, & Z''_{252} &= x_1^7 x_2^2 x_3^9 x_4^7 x_5^2, & Z''_{253} &= x_1^7 x_2^2 x_3^{11} x_4^6 x_5^6, \\
Z''_{254} &= x_1^7 x_2^2 x_3^{11} x_4^5 x_5^2, & Z''_{255} &= x_1^7 x_2^2 x_3^{15} x_4^2 x_5^2, & Z''_{256} &= x_1^7 x_2^{11} x_3^3 x_4^6 x_5^7, & Z''_{257} &= x_1^7 x_2^{11} x_3^3 x_4^7 x_5^6, \\
Z''_{258} &= x_1^7 x_2^{11} x_3^3 x_4^6 x_5^6, & Z''_{259} &= x_1^7 x_2^{11} x_3^5 x_4^7 x_5^7, & Z''_{260} &= x_1^7 x_2^{11} x_3^5 x_4^6 x_5^6, & Z''_{261} &= x_1^7 x_2^{11} x_3^5 x_4^6 x_5^3, \\
Z''_{262} &= x_1^7 x_2^{11} x_3^5 x_4^2 x_5^2, & Z''_{263} &= x_1^7 x_2^{11} x_3^7 x_4^6 x_5^6, & Z''_{264} &= x_1^7 x_2^{11} x_3^7 x_4^5 x_5^2, & Z''_{265} &= x_1^7 x_2^{15} x_3^2 x_4^2 x_5^7, \\
Z''_{266} &= x_1^7 x_2^{15} x_3^3 x_4^6 x_5^6, & Z''_{267} &= x_1^7 x_2^{15} x_3^6 x_4^3 x_5^3, & Z''_{268} &= x_1^7 x_2^{15} x_3^7 x_4^2 x_5^2, & Z''_{269} &= x_1^7 x_2^{15} x_3^3 x_4^6 x_5^6, \\
Z''_{270} &= x_1^7 x_2^{15} x_3^3 x_4^4 x_5^5, & Z''_{271} &= x_1^7 x_2^{15} x_3^5 x_4^2 x_5^2, & Z''_{272} &= x_1^7 x_2^{15} x_3^7 x_4^2 x_5^2, & Z''_{273} &= x_1^{15} x_2^2 x_3^2 x_4^7 x_5^7, \\
Z''_{274} &= x_1^{15} x_2^2 x_3^6 x_4^7 x_5^5, & Z''_{275} &= x_1^{15} x_2^2 x_3^7 x_4^6 x_5^6, & Z''_{276} &= x_1^{15} x_2^2 x_3^6 x_4^3 x_5^7, & Z''_{277} &= x_1^{15} x_2^2 x_3^6 x_4^7 x_5^3, \\
Z''_{278} &= x_1^{15} x_2^2 x_3^7 x_4^2 x_5^7, & Z''_{279} &= x_1^{15} x_2^2 x_3^7 x_4^3 x_5^6, & Z''_{280} &= x_1^{15} x_2^2 x_3^7 x_4^6 x_5^3, & Z''_{281} &= x_1^{15} x_2^2 x_3^7 x_4^7 x_5^2, \\
Z''_{282} &= x_1^{15} x_2^2 x_3^6 x_4^7 x_5^5, & Z''_{283} &= x_1^{15} x_2^2 x_3^7 x_4^6 x_5^6, & Z''_{284} &= x_1^{15} x_2^2 x_3^3 x_4^4 x_5^7, & Z''_{285} &= x_1^{15} x_2^2 x_3^3 x_4^5 x_5^6, \\
Z''_{286} &= x_1^{15} x_2^2 x_3^3 x_4^7 x_5^4, & Z''_{287} &= x_1^{15} x_2^2 x_3^5 x_4^2 x_5^7, & Z''_{288} &= x_1^{15} x_2^2 x_3^5 x_4^3 x_5^6, & Z''_{289} &= x_1^{15} x_2^2 x_3^5 x_4^6 x_5^3, \\
Z''_{290} &= x_1^{15} x_2^2 x_3^5 x_4^2 x_5^2, & Z''_{291} &= x_1^{15} x_2^2 x_3^7 x_4^6 x_5^6, & Z''_{292} &= x_1^{15} x_2^2 x_3^7 x_4^3 x_5^4, & Z''_{293} &= x_1^{15} x_2^2 x_3^7 x_4^5 x_5^2, \\
Z''_{294} &= x_1^{15} x_2^2 x_3^7 x_4^2 x_5^7, & Z''_{295} &= x_1^{15} x_2^2 x_3^3 x_4^5 x_5^6, & Z''_{296} &= x_1^{15} x_2^2 x_3^6 x_4^3 x_5^3, & Z''_{297} &= x_1^{15} x_2^2 x_3^7 x_4^2 x_5^2, \\
Z''_{298} &= x_1^{15} x_2^2 x_3^3 x_4^6 x_5^5, & Z''_{299} &= x_1^{15} x_2^2 x_3^3 x_4^5 x_5^4, & Z''_{300} &= x_1^{15} x_2^2 x_3^5 x_4^2 x_5^2, & Z''_{301} &= x_1^{15} x_2^2 x_3^7 x_4^2 x_5^2, \\
Z''_{302} &= x_1^3 x_2^7 x_3^7 x_4^{11} x_5^4, & Z''_{303} &= x_1^3 x_2^7 x_3^{11} x_4^4 x_5^7, & Z''_{304} &= x_1^3 x_2^7 x_3^7 x_4^8 x_5^7, & Z''_{305} &= x_1^3 x_2^7 x_3^{11} x_4^7 x_5^4, \\
Z''_{306} &= x_1^7 x_2^3 x_3^7 x_4^{11} x_5^4, & Z''_{307} &= x_1^7 x_2^3 x_3^{11} x_4^4 x_5^7, & Z''_{308} &= x_1^7 x_2^3 x_3^{11} x_4^3 x_5^4, & Z''_{309} &= x_1^7 x_2^{11} x_3^7 x_4^3 x_5^4, \\
Z''_{310} &= x_1^7 x_2^3 x_3^3 x_4^6 x_5^6, & Z''_{311} &= x_1^7 x_2^3 x_3^{11} x_4^4 x_5^7, & Z''_{312} &= x_1^7 x_2^{11} x_3^3 x_4^4 x_5^7, & Z''_{313} &= x_1^7 x_2^9 x_3^3 x_4^6 x_5^7, \\
Z''_{314} &= x_1^7 x_2^3 x_3^8 x_4^7 x_5^7, & Z''_{315} &= x_1^7 x_2^3 x_3^8 x_4^4 x_5^7, & Z''_{316} &= x_1^7 x_2^3 x_3^7 x_4^8 x_5^7, & Z''_{317} &= x_1^7 x_2^3 x_3^{11} x_4^7 x_5^4, \\
Z''_{318} &= x_1^7 x_2^{11} x_3^3 x_4^4 x_5^4, & Z''_{319} &= x_1^7 x_2^9 x_3^3 x_4^4 x_5^6, & Z''_{320} &= x_1^7 x_2^8 x_3^7 x_4^3 x_5^3.
\end{aligned}$$

3.9. An algorithm in MAGMA

In this subsection, to verify the dimension of the space $Q^{\otimes 5}$ in degree $N = 31$ and degree $N = 32$, we provide an algorithm in MAGMA, which is established as follows:

N := 31;

```

R<x1,x2,x3,x4,x5> := PolynomialRing(GF(2),5);
P<t> := PolynomialRing(R);
Sqhom := hom<R->P | [x + t*x^2 : x in [x1,x2,x3,x4,x5]]>;
function Sq(j,x)

```

```

    c := Coefficients(Sqhom(x));
    if j+1 gt #c then
        return R!0;
    else
        return c[j+1];
    end if;
end function;
M := MonomialsOfDegree(R,N);
V := VectorSpace(GF(2),#M);
function MtoV(m)
    if IsZero(m) then
        return V!0;
    else
        cv := [Index(M,mm) : mm in Terms(m)];
        return CharacteristicVector(V,cv);
    end if;
end function;
M1 := MonomialsOfDegree(R,N-1);
M2 := MonomialsOfDegree(R,N-2);
M4 := MonomialsOfDegree(R,N-4);
M8 := MonomialsOfDegree(R,N-8);
M16 := MonomialsOfDegree(R,N-16);
print [#m : m in [M1,M2,M4,M8, M16]];

> output: [46376, 40920, 31465, 17550, 3876]

S1 := [Sq(1,x) : x in M1];
print "S1";

H1 := sub<V | [MtoV(x) : x in S1]>;
print Dimension(H1);

> output: 24615

S2 := [Sq(2,x) : x in M2];
print "S2";

H2 := sub<V | [MtoV(x) : x in S2]>;
print Dimension(H2);

> output: 28665

H := H1 + H2;
print Dimension(H);

> output: 43334

S4 := [Sq(4,x) : x in M4];
print "S4";

H4 := sub<V | [MtoV(x): x in S4]>;
print Dimension(H4);

> output: 26520

H := H + H4;
print Dimension(H);

> output: 49530

S8 := [Sq(8,x) : x in M8];
print "S8";

H8 := sub<V | [MtoV(x) : x in S8]>;
print Dimension(H8);

> output: 15900

```

```

H := H + H8;
print Dimension(H);
> output: 51494
S16 := [Sq(16,x) : x in M16];
print "S16";

H16 := sub<V | [MtoV(x) : x in S16]>;
print Dimension(H16);
> output: 0
H := H + H16;
print Dimension(H);
> output: 51494
print "\n", Dimension(V);
> output: 52360
print Dimension(H);
> output: 51494
print Dimension(V) - Dimension(H);

```

The calculations indicated that

$$\dim Q_{31}^{\otimes 5} = \dim V - \dim H = 52360 - 51494 = 866.$$

Replacing $N = 31$ by $N = 32$ in the above algorithm, we obtain

$$\dim Q_{32}^{\otimes 5} = \dim V - \dim H = 58905 - 57901 = 1004.$$

Remark. To compute the hit monomials and the inadmissible monomials, we only need to consider the effects of the Steenrod squaring operations $Sq^j : \mathbb{P}_{N-j}^{\otimes 5} \rightarrow \mathbb{P}_N^{\otimes 5}$ for $j = 2^t$, $t \geq 0$, $2j \leq N$. This can be explained as follows: Because the set of all these squares are a minimal generating set for $\mathcal{A}_2$ as an algebra over $\mathbb{Z}_2$. Further, as well known, for any $X \in \mathbb{P}_{N-j}^{\otimes 5}$, $Sq^{2^t}(X) = 0$ if $\deg(X) = N - j < 2^t$. And therefore we need only to compute the action of the squares Sq^{2^t} on $\mathbb{P}_N^{\otimes 5}$ such that $2^t \leq \deg(X)$. More importantly, by using Kameko's homomorphism and changing the number of variables in the polynomial ring " $R \langle x_1, x_2, \dots, x_5 \rangle$ " and degree " N " in the above algorithm, we can verify all the results of Peterson [20], Kameko [13] and Sum [34] on the dimension of the space (1.2) for $t \leq 4$.

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