GLOBALLY CONVERGENT SERIES FOR THE DIRICHLET ETA AND BETA FUNCTIONS

SUMIT KUMAR JHA

ABSTRACT. We derive the the following globally convergent series for the Dirichlet eta and beta functions

$$\eta(s) = \sum_{k=0}^{\infty} \frac{2^s}{2^{\frac{k+1}{2}}} \cos\left(\frac{(k+1)\pi}{4}\right) \sum_{m=0}^k \binom{k}{m} \frac{(-1)^m}{(m+1)^s},$$

$$\beta(s) = \sum_{k=0}^{\infty} \frac{1}{2^{\frac{k+1}{2}}} \sin\left(\frac{(k+1)\pi}{4}\right) \sum_{m=0}^k \binom{k}{m} \frac{(-1)^m}{(m+1)^s}.$$

1. MAIN RESULTS

Let $\eta(s)$ represent the Dirichlet eta function. H. Hasse ([1], [2]) proved that

$$\eta(s) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(k+1)^s}$$

We prove the following.

Theorem 1. *Let $\eta(s)$ and $\beta(s)$ represent the Dirichlet eta function, and the Dirichlet beta function, respectively. Then we have*

$$\eta(s) = \sum_{k=0}^{\infty} \frac{2^s}{2^{\frac{k+1}{2}}} \cos\left(\frac{(k+1)\pi}{4}\right) \sum_{m=0}^k \binom{k}{m} \frac{(-1)^m}{(m+1)^s}, \quad (1)$$

$$\beta(s) = \sum_{k=0}^{\infty} \frac{1}{2^{\frac{k+1}{2}}} \sin\left(\frac{(k+1)\pi}{4}\right) \sum_{m=0}^k \binom{k}{m} \frac{(-1)^m}{(m+1)^s}. \quad (2)$$

Proof. The polylogarithm function, $\text{Li}_s(z)$, has the following series representation [3, Theorem 2.1]

$$\text{Li}_s(z) = \sum_{k=0}^{\infty} \left(\frac{-z}{1-z}\right)^{k+1} \sum_{m=0}^k \binom{k}{m} \frac{(-1)^{m+1}}{(m+1)^s}, \quad (3)$$

for all $s, z \in \mathbb{C}$ with $\Re(z) < 1/2$. Now, recalling that

$$\text{Li}_s(-i) = -2^{-s} \eta(s) - i \beta(s),$$

we can derive equations (1) and (2) by taking real and imaginary parts in the series expression of $\text{Li}_s(-i)$, respectively. $\square$

2010 Mathematics Subject Classification. 11M06.

Key words and phrases. Dirichlet eta function, Dirichlet beta function.

REFERENCES

1. Hasse, Helmut (1930). *Ein Summierungsverfahren für die Riemannsche ζ -Reihe* [A summation method for the Riemann ζ series]. *Mathematische Zeitschrift* (in German). 32 (1): 458-464. doi:10.1007/BF01194645
2. Sondow, Jonathan (1994). *Analytic continuation of Riemann's zeta function and values at negative integers via Euler's transformation of series*. *Proceedings of the American Mathematical Society*. 120 (2): 421-424. doi:10.1090/S0002-9939-1994-1172954-7.
3. J. Guillera, J. Sondow, *Double integrals and infinite products for some classical constants via analytic continuations of Lerch's transcendent*, *J. Ramanujan J* (2008) 16: 247. <https://doi.org/10.1007/s11139-007-9102-0>

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY, HYDERABAD, INDIA

E-mail address: kumarjha.sumit@research.iiit.ac.in