A SUM IN TERMS OF THE DIRICHLET BETA FUNCTION

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ABSTRACT. In this note, we derive the following sum

$$\beta(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^s} = - \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_s} \frac{2^{i_s-1}}{i_s \binom{2i_s}{i_s}} \cdot \frac{1}{(1+2i_1)(1+2i_2) \cdots (1+2i_s)},$$

where s is any natural number, and the summation is over all s -tuples $(i_1, i_2, \dots, i_s)$ of natural numbers satisfying $1 \leq i_1 \leq i_2 \leq \dots \leq i_s$ for fixed s .

1. MAIN RESULT

Theorem 1. *Let s be any natural number. Then we have*

$$\beta(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^s} = - \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_s} \frac{2^{i_s-1}}{i_s \binom{2i_s}{i_s}} \cdot \frac{1}{(1+2i_1)(1+2i_2) \cdots (1+2i_s)}. \quad (1)$$

Proof. To derive equation (1), we use the following result from [1]

$$\begin{aligned} \text{Li}_s^\alpha(w) &= \sum_{n=1}^{\infty} \frac{w^n}{(\alpha+n)^s} \\ &= - \sum_{1 \leq i_1 \leq \dots \leq i_s} \frac{(i_s-1)!}{(\alpha+1)(\alpha+2) \cdots (\alpha+i_s)} \cdot \frac{1}{(\alpha+i_1)(\alpha+i_2) \cdots (\alpha+i_{s-1})} \cdot \left(\frac{-w}{1-w} \right)^{i_s} \end{aligned}$$

Letting $\alpha = 1/2$ and $w = -1$ in the above we get (1) at once. □

2. SPECIAL CASES

Letting $s = 1$ in equation (1) we get

$$\frac{\pi}{4} = 1 - \sum_{i=1}^{\infty} \frac{2^{i-1}}{i \binom{2i}{i}} \cdot \frac{1}{(1+2i)}.$$

Remark 1. *Some other such type of sums have also been obtained in [2].*

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