COMPLETE MONOTONICITY OF A DIFFERENCE DEFINED BY FOUR DERIVATIVES OF A FUNCTION CONTAINING TRIGAMMA FUNCTION

FENG QI

Dedicated to people facing and battling COVID-19

ABSTRACT. In the paper, by virtue of convolution theorem for the Laplace transforms, logarithmic convexity of the gamma function, Bernstein's theorem for completely monotonic functions, and other techniques, the author finds necessary and sufficient conditions for a difference defined by four derivatives of a function containing trigamma function to be completely monotonic. Moreover, by virtue of Čebyšev integral inequality, the author presents logarithmic convexity of the sequence of polygamma functions.

CONTENTS

1.	Preliminaries and motivations	1
2.	Lemmas	4
3.	Necessary and sufficient conditions of complete monotonicity	5
4.	Logarithmic convexity of the sequence of polygamma functions	8
5.	Remarks	9
	References	9

1. PRELIMINARIES AND MOTIVATIONS

In the literature [1, Section 6.4], the function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

and its logarithmic derivative

$$\psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$$

are called Euler's gamma function and digamma function respectively. Further, the functions $\psi'(z)$, $\psi''(z)$, $\psi'''(z)$, and $\psi^{(4)}(z)$ are known as trigamma, tetragamma, pentagamma, and hexagamma functions respectively. All the derivatives $\psi^{(k)}(z)$

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for $k \in \{0\} \cup \mathbb{N}$ are known as polygamma functions. There have been a number of literature dedicated to studying of the gamma and polygamma functions. See the papers [2, 3, 4, 7, 8, 13, 15, 20, 21, 22, 24, 26, 32, 33] and closely related references therein.

Recall from Chapter XIII in [11], Chapter 1 in [27], and Chapter IV in [30] that, if a function $f(x)$ on an interval I has derivatives of all orders on I and satisfies $(-1)^n f^{(n)}(x) \geq 0$ for $x \in I$ and $n \in \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers, then we call $f(x)$ a completely monotonic function on I . There have been a number of literature dedicated to studying of completely monotonic functions and their degrees. See the papers [2, 3, 4, 5, 7, 8, 13, 15, 20, 21, 22, 26, 32, 33] and closely related references therein.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{R}^n$. A n -tuple α is said to strictly majorize β (in symbols $\alpha \succ \beta$) if $(\alpha_{[1]}, \alpha_{[2]}, \dots, \alpha_{[n]}) \neq (\beta_{[1]}, \beta_{[2]}, \dots, \beta_{[n]})$, $\sum_{i=1}^k \alpha_{[i]} \geq \sum_{i=1}^k \beta_{[i]}$ for $1 \leq k \leq n-1$, and $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$, where $\alpha_{[1]} \geq \alpha_{[2]} \geq \dots \geq \alpha_{[n]}$ and $\beta_{[1]} \geq \beta_{[2]} \geq \dots \geq \beta_{[n]}$ are rearrangements of α and β in a descending order. A real-valued function ϕ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be Schur-convex on $\mathcal{A}$ if $\mathbf{x} \prec \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathcal{A}$ means $\phi(\mathbf{x}) < \phi(\mathbf{y})$. See [9, p. 8, Definition A.1] [9, p. 80, Definition A.1]. There have been a lot of literature such as the papers [6, 26, 28, 31, 34, 35] dedicated to investigation of Schur-convexity.

Let $\Phi(x) = x\psi'(x) - 1 = x[\psi'(x) - \frac{1}{x}]$ on $(0, \infty)$. In [17, Theorem 4.1] and [20, Theorem 4], the author gave the following necessary and sufficient conditions and double inequality:

- (1) if and only if $\alpha \geq 2$, the function $\mathfrak{H}_\alpha(x) = \Phi'(x) + \alpha\Phi^2(x)$ is completely monotonic on $(0, \infty)$;
- (2) if and only if $\alpha \leq 1$, the function $-\mathfrak{H}_\alpha(x)$ is completely monotonic on $(0, \infty)$;
- (3) the double inequality $-2 < \frac{\Phi'(x)}{\Phi^2(x)} < -1$ is valid and sharp in the sense that the lower and upper bounds -2 and -1 cannot be replaced by any bigger and smaller ones respectively.

In [17, Theorem 1.1], the author found the following necessary and sufficient conditions and limits:

- (1) if and only if $\beta \geq 2$, the function $H_\beta(x) = \frac{\Phi'(x)}{\Phi^\beta(x)}$ is decreasing on $(0, \infty)$, with the limits

$$\lim_{x \rightarrow 0^+} H_\beta(x) = \begin{cases} -1, & \beta = 2 \\ 0, & \beta > 2 \end{cases} \quad \text{and} \quad \lim_{x \rightarrow \infty} H_\beta(x) = \begin{cases} -2, & \beta = 2 \\ -\infty, & \beta > 2; \end{cases}$$

- (2) if $\beta \leq 1$, the function $H_\beta(x)$ is increasing on $(0, \infty)$, with the limits

$$H_\beta(x) \rightarrow \begin{cases} -\infty, & x \rightarrow 0^+ \\ 0, & x \rightarrow \infty. \end{cases}$$

For $k \in \{0\} \cup \mathbb{N}$ and $\lambda_k, \mu_k \in \mathbb{R}$, let

$$\mathfrak{J}_{k, \lambda_k}(x) = \Phi^{(2k+1)}(x) + \lambda_k [\Phi^{(k)}(x)]^2 \quad \text{and} \quad J_{k, \mu_k}(x) = \frac{\Phi^{(2k+1)}(x)}{[(-1)^k \Phi^{(k)}(x)]^{\mu_k}}$$

on $(0, \infty)$. In [12, Theorems 3.1 and 4.1], the author gave the following necessary and sufficient conditions, limits, and double inequality:

- (1) if and only if $\lambda_k \geq \frac{(2k+2)!}{k!(k+1)!}$, the function $\mathfrak{J}_{k,\lambda_k}(x)$ is completely monotonic on $(0, \infty)$;
- (2) if and only if $\lambda_k \leq \frac{1}{2} \frac{(2k+2)!}{k!(k+1)!}$, the function $-\mathfrak{J}_{k,\lambda_k}(x)$ is completely monotonic on $(0, \infty)$;
- (3) if and only if $\mu_k \geq 2$, the function $J_{k,\mu_k}(x)$ is decreasing on $(0, \infty)$, with the limits

$$\lim_{x \rightarrow 0^+} J_{k,\mu_k}(x) = \begin{cases} -\frac{1}{2} \frac{(2k+2)!}{k!(k+1)!}, & \mu_k = 2 \\ 0, & \mu_k > 2 \end{cases}$$

and

$$\lim_{x \rightarrow \infty} J_{k,\mu_k}(x) = \begin{cases} -\frac{(2k+2)!}{k!(k+1)!}, & \mu_k = 2 \\ -\infty, & \mu_k > 2; \end{cases}$$

- (4) if $\mu_k \leq 1$, the function $J_{k,\mu_k}(x)$ is increasing on $(0, \infty)$, with the limits

$$J_{k,\mu_k}(x) \rightarrow \begin{cases} -\infty, & x \rightarrow 0^+ \\ 0, & x \rightarrow \infty; \end{cases}$$

- (5) the double inequality

$$-\frac{(2k+2)!}{k!(k+1)!} < \frac{\Phi^{(2k+1)}(x)}{[\Phi^{(k)}(x)]^2} < -\frac{1}{2} \frac{(2k+2)!}{k!(k+1)!}$$

is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.

For $m, n \in \{0\} \cup \mathbb{N}$ and $\omega_{m,n} \in \mathbb{R}$, let

$$Y_{m,n}(x) = \frac{\Phi^{(m+n+1)}(x)}{\Phi^{(m)}(x)\Phi^{(n)}(x)}$$

and

$$\mathcal{Y}_{m,n;\omega_{m,n}}(x) = \Phi^{(m+n+1)}(x) + \omega_{m,n} \Phi^{(m)}(x)\Phi^{(n)}(x).$$

In [23, Theorems 3.1 and 4.1], the following necessary and sufficient conditions, monotonicity, complete monotonicity, and double inequality were established:

- (1) the function $Y_{m,n}(x)$ is decreasing in x from $(0, \infty)$ onto the open interval

$$\left(-\frac{2(m+n+1)!}{m!n!}, -\frac{(m+n+1)!}{m!n!} \right);$$

- (2) if and only if $\omega_{m,n} \leq \frac{(m+n+1)!}{m!n!}$, the function $(-1)^{m+n+1} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$ is completely monotonic on $(0, \infty)$;
- (3) if and only if $\omega_{m,n} \geq \frac{2(m+n+1)!}{m!n!}$, the function $(-1)^{m+n} \mathcal{Y}_{m,n;\omega_{m,n}}(x)$ is completely monotonic on $(0, \infty)$;
- (4) the double inequality

$$-\frac{2(m+n+1)!}{m!n!} < Y_{m,n}(x) < -\frac{(m+n+1)!}{m!n!}$$

is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.

In this paper, we would like to consider monotonicity of the function

$$\mathbf{Y}_{i,j;\ell,m}(x) = \frac{\Phi^{(i)}(x)\Phi^{(j)}(x)}{\Phi^{(\ell)}(x)\Phi^{(m)}(x)}$$

and complete monotonicity of the function

$$\mathfrak{Y}_{i,j;\ell,m;\Omega_{i,j;\ell,m}}(x) = (-1)^{i+j}\Phi^{(i)}(x)\Phi^{(j)}(x) - (-1)^{\ell+m}\Omega_{i,j;\ell,m}\Phi^{(\ell)}(x)\Phi^{(m)}(x) \quad (1.1)$$

on $(0, \infty)$, where $i, j, \ell, m \geq 0$ are integers such that $(i, j) \succ (\ell, m)$. Figure 1 plotted by the software MATHEMATICA hints that the function $\mathbf{Y}_{17,11;15,13}(x)$ is not monotonic in $x \in (0, \infty)$. Therefore, in this paper, we will only consider the

FIGURE 1. The graph of the function $\mathbf{Y}_{17,11;15,13}(x)$ on $(0, 9)$

functions $\pm\mathfrak{Y}_{i,j;\ell,m;\Omega_{i,j;\ell,m}}(x)$ and find necessary and sufficient conditions on $\Omega_{i,j;\ell,m}$ for $\pm\mathfrak{Y}_{i,j;\ell,m;\Omega_{i,j;\ell,m}}(x)$ to be completely monotonic on $(0, \infty)$.

Moreover, we will also present that the sequence of polygamma functions $\psi^{(n)}(x)$ for $n \in \mathbb{N}$ is logarithmic convex.

2. LEMMAS

The following lemmas are necessary in this paper.

Lemma 2.1 ([17, Lemma 2.3]). *Let*

$$h(t) = \begin{cases} \frac{e^t(e^t - 1 - t)}{(e^t - 1)^2}, & t \neq 0; \\ \frac{1}{2}, & t = 0. \end{cases}$$

For any fixed $t > 0$, the function $h(st)h((1-s)t)$ is increasing in $s \in (0, \frac{1}{2})$.

Lemma 2.2 (Convolution theorem for the Laplace transforms [30, pp. 91–92]). *Let $f_k(t)$ for $k = 1, 2$ be piecewise continuous in arbitrary finite intervals included in $(0, \infty)$. If there exist some constants $M_k > 0$ and $c_k \geq 0$ such that $|f_k(t)| \leq M_k e^{c_k t}$ for $k = 1, 2$, then*

$$\int_0^\infty \left[\int_0^t f_1(u)f_2(t-u)du \right] e^{-st} dt = \int_0^\infty f_1(u)e^{-su} du \int_0^\infty f_2(v)e^{-sv} dv.$$

Lemma 2.3 ([14, Lemma 2.6]). *For $m, n, p, q \in \mathbb{N}$ such that $(p, q) \succ (m, n)$, the function*

$$\frac{s^{m-1}(1-s)^{n-1} + (1-s)^{m-1}s^{n-1}}{s^{p-1}(1-s)^{q-1} + (1-s)^{p-1}s^{q-1}}$$

is increasing in s from $(0, \frac{1}{2})$ onto $(0, 1)$.

Lemma 2.4. *For $i, j, \ell, m \in \{0\} \cup \mathbb{N}$ with $(i, j) \succ (\ell, m)$, the inequality $i!j! > \ell!m!$ is valid.*

Proof. It is well-known that the function $\Gamma(x)$ is logarithmically convex on $(0, \infty)$. See [18, Corollary 1] and [29, p. 42, Section 3.1.2]. In [9, p. 92, C.1.a (i)], it is stated that, if $g(x)$ is convex on an interval I , then the function $\phi(\mathbf{x}) = \sum_{k=1}^n g(x_k)$ is Schur-convex on I^n , where $\mathbf{x} = (x_1, x_2, \dots, x_n) \in I^n$. Hence, taking $g(x) = \ln \Gamma(x)$, $n = 2$, and $I = (0, \infty)$ yields that, when $(i, j) \succ (\ell, m)$ and $i, j, \ell, m \in \mathbb{N}$, we have

$$\ln \Gamma(i) + \ln \Gamma(j) > \ln \Gamma(\ell) + \ln \Gamma(m),$$

that is,

$$(i-1)!(j-1)! = \Gamma(i)\Gamma(j) > \Gamma(\ell)\Gamma(m) = (\ell-1)!(m-1)!.$$

The proof of Lemma 2.4 is complete. $\square$

Lemma 2.5 (Bernstein's theorem [30, p. 161, Theorem 12b]). *A function $f(x)$ is completely monotonic on $(0, \infty)$ if and only if*

$$f(x) = \int_0^\infty e^{-xt} d\sigma(t), \quad x \in (0, \infty), \quad (2.1)$$

where $\sigma(s)$ is non-decreasing and the integral in (2.1) converges for $x \in (0, \infty)$.

3. NECESSARY AND SUFFICIENT CONDITIONS OF COMPLETE MONOTONICITY

In this section, we find necessary and sufficient conditions on $\Omega_{i,j;\ell,m}$ such that the functions $\pm \mathfrak{Y}_{i,j;\ell,m;\Omega_{i,j;\ell,m}}(x)$ defined by (1.1) are completely monotonic.

Theorem 3.1. *For $i, j, \ell, m \geq 0$ are integers such that $(i, j) \succ (\ell, m)$,*

- (1) *if $\Omega_{i,j;\ell,m} \leq 1$, the function $\mathfrak{Y}_{i,j;\ell,m;\Omega_{i,j;\ell,m}}(x)$ defined in (1.1) is completely monotonic on $(0, \infty)$;*
- (2) *if and only if $\Omega_{i,j;\ell,m} \geq \frac{i!j!}{\ell!m!}$, the function $-\mathfrak{Y}_{i,j;\ell,m;\Omega_{i,j;\ell,m}}(x)$ is completely monotonic on $(0, \infty)$;*
- (3) *the double inequality*

$$1 < \frac{\Phi^{(i)}(x)\Phi^{(j)}(x)}{\Phi^{(\ell)}(x)\Phi^{(m)}(x)} < \frac{i!j!}{\ell!m!} \quad (3.1)$$

is valid on $(0, \infty)$ and the right hand side inequality is sharp in the sense that the number $\frac{i!j!}{\ell!m!}$ can not be replaced by any smaller one.

Proof. In the proof of [20, Theorem 4], the author derived an integral representation

$$\Phi(x) = \int_0^\infty h(t)e^{-xt} dt,$$

where $h(t)$ is defined in Lemma 2.1. By virtue of Lemma 2.2, we acquire

$$\mathfrak{Y}_{i,j;\ell,m;\Omega_{i,j;\ell,m}}(x) = \int_0^\infty t^i h(t)e^{-xt} dt \int_0^\infty t^j h(t)e^{-xt} dt$$

$$\begin{aligned}
& -\Omega_{i,j;\ell,m} \int_0^\infty t^\ell h(t) e^{-xt} dt \int_0^\infty t^m h(t) e^{-xt} dt \\
&= \int_0^\infty \left[\int_0^t u^i (t-u)^j h(u) h(t-u) du \right] e^{-xt} dt \\
&\quad - \Omega_{i,j;\ell,m} \int_0^\infty \left[\int_0^t u^\ell (t-u)^m h(u) h(t-u) du \right] e^{-xt} dt \\
&= \int_0^\infty \left[\frac{\int_0^t u^i (t-u)^j h(u) h(t-u) du}{\int_0^t u^\ell (t-u)^m h(u) h(t-u) du} - \Omega_{i,j;\ell,m} \right] \\
&\quad \times \left[\int_0^t u^\ell (t-u)^m h(u) h(t-u) du \right] e^{-xt} dt \\
&= \int_0^\infty \left[\frac{\int_0^1 s^i (1-s)^j h(st) h((1-s)t) ds}{\int_0^1 s^\ell (1-s)^m h(st) h((1-s)t) ds} - \Omega_{i,j;\ell,m} \right] \\
&\quad \times \left[\int_0^t u^\ell (t-u)^m h(u) h(t-u) du \right] e^{-xt} dt \\
&= \int_0^\infty \left[\frac{\int_0^{1/2} [s^i (1-s)^j + s^j (1-s)^i] h(st) h((1-s)t) ds}{\int_0^{1/2} [s^\ell (1-s)^m + s^m (1-s)^\ell] h(st) h((1-s)t) ds} \right. \\
&\quad \left. - \Omega_{i,j;\ell,m} \right] \left[\int_0^t u^\ell (t-u)^m h(u) h(t-u) du \right] e^{-xt} dt.
\end{aligned}$$

By Lemma 2.3, we obtain that the double inequality

$$1 < \frac{s^i (1-s)^j + (1-s)^j s^i}{s^\ell (1-s)^m + (1-s)^m s^\ell} < \infty \quad (3.2)$$

is valid and sharp for $s \in (0, \frac{1}{2})$ and $(i, j) \succ (\ell, m)$. This means that

$$\frac{\int_0^{1/2} [s^i (1-s)^j + s^j (1-s)^i] h(st) h((1-s)t) ds}{\int_0^{1/2} [s^\ell (1-s)^m + s^m (1-s)^\ell] h(st) h((1-s)t) ds} > 1$$

for $t \in (0, \infty)$ and $(i, j) \succ (\ell, m)$. Therefore, when $\Omega_{i,j;\ell,m} \leq 1$ and $(i, j) \succ (\ell, m)$, the function $\mathfrak{V}_{i,j;\ell,m;\Omega_{i,j;\ell,m}}(x)$ is completely monotonic on $(0, \infty)$.

It is easy to see that

$$\begin{aligned}
& \lim_{t \rightarrow 0^+} \frac{\int_0^{1/2} [s^i (1-s)^j + s^j (1-s)^i] h(st) h((1-s)t) ds}{\int_0^{1/2} [s^\ell (1-s)^m + s^m (1-s)^\ell] h(st) h((1-s)t) ds} \\
&= \lim_{t \rightarrow \infty} \frac{\int_0^{1/2} [s^i (1-s)^j + s^j (1-s)^i] h(st) h((1-s)t) ds}{\int_0^{1/2} [s^\ell (1-s)^m + s^m (1-s)^\ell] h(st) h((1-s)t) ds} \\
&= \frac{\int_0^{1/2} [s^i (1-s)^j + s^j (1-s)^i] ds}{\int_0^{1/2} [s^\ell (1-s)^m + s^m (1-s)^\ell] ds} = \frac{B(i+1, j+1)}{B(\ell+1, m+1)} = \frac{i!j!}{\ell!m!}.
\end{aligned}$$

Let

$$S_{i,j;\ell,m}(t) = \int_0^{1/2} [s^i (1-s)^j + s^j (1-s)^i] h(st) h((1-s)t) ds$$

$$\begin{aligned}
& - \frac{i!j!}{\ell!m!} \int_0^{1/2} [s^\ell(1-s)^m + s^m(1-s)^\ell] h(st)h((1-s)t)ds \\
& = \int_0^{1/2} \left([s^i(1-s)^j + s^j(1-s)^i] \right. \\
& \quad \left. - \frac{i!j!}{\ell!m!} [s^\ell(1-s)^m + s^m(1-s)^\ell] \right) h(st)h((1-s)t)ds \\
& = \int_0^{1/2} \left[\frac{s^i(1-s)^j + s^j(1-s)^i}{s^\ell(1-s)^m + s^m(1-s)^\ell} - \frac{i!j!}{\ell!m!} \right] \\
& \quad \times [s^\ell(1-s)^m + s^m(1-s)^\ell] h(st)h((1-s)t)ds
\end{aligned}$$

for $t \in (0, \infty)$ and $(i, j) \succ (\ell, m)$. By Lemma 2.3, Lemma 2.4, and the sharp inequality (3.2), we arrive at that the function

$$\frac{s^i(1-s)^j + s^j(1-s)^i}{s^\ell(1-s)^m + s^m(1-s)^\ell} - \frac{i!j!}{\ell!m!}$$

is decreasing in $s \in (0, \frac{1}{2})$ and has a unique zero $s_0 \in (0, \frac{1}{2})$ for $(i, j) \succ (\ell, m)$. As a result, utilizing Lemma 2.1, we have

$$\begin{aligned}
S_{i,j;\ell,m}(t) & = \left(\int_0^{s_0} + \int_{s_0}^{1/2} \right) \left[\frac{s^i(1-s)^j + s^j(1-s)^i}{s^\ell(1-s)^m + s^m(1-s)^\ell} - \frac{i!j!}{\ell!m!} \right] \\
& \quad \times [s^\ell(1-s)^m + s^m(1-s)^\ell] h(st)h((1-s)t)ds \\
& < h(s_0t)h((1-s_0)t) \int_0^{1/2} \left[\frac{s^i(1-s)^j + s^j(1-s)^i}{s^\ell(1-s)^m + s^m(1-s)^\ell} - \frac{i!j!}{\ell!m!} \right] \\
& \quad \times [s^\ell(1-s)^m + s^m(1-s)^\ell] ds \\
& = h(s_0t)h((1-s_0)t) \int_0^{1/2} \left(s^i(1-s)^j + s^j(1-s)^i \right. \\
& \quad \left. - \frac{i!j!}{\ell!m!} [s^\ell(1-s)^m + s^m(1-s)^\ell] \right) ds \\
& = h(s_0t)h((1-s_0)t) \left(\int_0^{1/2} [s^i(1-s)^j + s^j(1-s)^i] ds \right. \\
& \quad \left. - \frac{i!j!}{\ell!m!} \int_0^{1/2} [s^\ell(1-s)^m + s^m(1-s)^\ell] ds \right) \\
& = h(s_0t)h((1-s_0)t) \left[B(i+1, j+1) - \frac{i!j!}{\ell!m!} B(\ell+1, m+1) \right] \\
& = h(s_0t)h((1-s_0)t) \left[\frac{i!j!}{(i+j+1)!} - \frac{i!j!}{\ell!m!} \frac{\ell!m!}{(\ell+m+1)!} \right] \\
& = 0.
\end{aligned}$$

Accordingly, the inequality

$$\frac{\int_0^{1/2} [s^i(1-s)^j + s^j(1-s)^i] h(st)h((1-s)t)ds}{\int_0^{1/2} [s^\ell(1-s)^m + s^m(1-s)^\ell] h(st)h((1-s)t)ds} < \frac{i!j!}{\ell!m!}$$

is valid and sharp for $t \in (0, \infty)$ and $(i, j) \succ (\ell, m)$. Therefore, by Lemma 2.5, for $(i, j) \succ (\ell, m)$, if and only if $\Omega_{i,j;\ell,m} \geq \frac{i!j!}{\ell!m!}$, the function $-\mathfrak{Y}_{i,j;\ell,m;\Omega_{i,j;\ell,m}}(x)$ is completely monotonic on $(0, \infty)$.

The double inequality (3.1) follows from complete monotonicity of the functions $\pm \mathfrak{Y}_{i,j;\ell,m;\Omega_{i,j;\ell,m}}(x)$ on $(0, \infty)$. The sharpness of the right hand side inequality in (3.1) follows from the limits

$$(-1)^k x^{k+1} \Phi^{(k)}(x) \rightarrow \begin{cases} k!, & x \rightarrow 0^+ \\ \frac{k!}{2}, & x \rightarrow \infty \end{cases}$$

in [12, Lemma 2.2], where $k \geq 0$. The proof of Theorem 3.1 is complete. $\square$

4. LOGARITHMIC CONVEXITY OF THE SEQUENCE OF POLYGAMMA FUNCTIONS

In this section, we present that the sequence of polygamma functions $\psi^{(n)}(x)$ for $n \in \mathbb{N}$ is logarithmic convex.

Theorem 4.1. *For fixed $s \in (0, \infty)$ and a variable $t \in (0, \infty)$, the function*

$$\Phi(s, t) = \int_0^\infty \frac{u^t}{1 - e^{-u}} e^{-su} du$$

is logarithmically convex in $t \in (0, \infty)$. Consequently, the sequence $(-1)^{n+1} \psi^{(n)}(x)$ is logarithmically convex in $n \in \mathbb{N}$ and

$$\psi^{(m-k)}(x) \psi^{(m+k)}(x) \geq [\psi^{(m)}(x)]^2$$

for $m > k > 0$.

Proof. Two functions $f(u)$ and $g(u)$ are called synchronous on an interval I if

$$[f(u) - f(v)][g(u) - g(v)] \geq 0, \quad x, y \in I.$$

The Čebyšev integral inequality [10, p. 40, Theorem 10] reads that

$$\int_a^b p(u) f(u) g(u) dx \int_a^b p(u) dx \geq \int_a^b p(u) f(u) dx \int_a^b p(u) g(u) dx, \quad (4.1)$$

where $f(u)$ and $g(u)$ are integrable and synchronous functions on (a, b) and $p(u)$ is a positive and integrable function on (a, b) . See also the paper [25] and closely related references therein.

Direct calculation yields

$$\frac{\partial \ln \Phi(s, t)}{\partial t} = \frac{\int_0^\infty \frac{u^t \ln u}{1 - e^{-u}} e^{-su} du}{\int_0^\infty \frac{u^t}{1 - e^{-u}} e^{-su} du}$$

and

$$\frac{\partial^2 \ln \Phi(s, t)}{\partial t^2} = \frac{\int_0^\infty \frac{u^t (\ln u)^2}{1 - e^{-u}} e^{-su} du \int_0^\infty \frac{u^t}{1 - e^{-u}} e^{-su} du - [\int_0^\infty \frac{u^t \ln u}{1 - e^{-u}} e^{-su} du]^2}{[\int_0^\infty \frac{u^t}{1 - e^{-u}} e^{-su} du]^2}.$$

Applying $f(u) = g(u) = \ln u$ and $p(u) = \frac{u^t}{1 - e^{-u}} e^{-su}$ on the interval $(a, b) = (0, \infty)$ to the inequality (4.1) leads to

$$\int_0^\infty \frac{u^t (\ln u)^2}{1 - e^{-u}} e^{-su} du \int_0^\infty \frac{u^t}{1 - e^{-u}} e^{-su} du \geq \left[\int_0^\infty \frac{u^t \ln u}{1 - e^{-u}} e^{-su} du \right]^2$$

which means $\frac{\partial^2 \ln \Phi(s,t)}{\partial t^2} \geq 0$. Hence, the function $\Phi(s,t)$ is logarithmically convex in $t \in (0, \infty)$. The proof of Theorem 4.1 is complete. $\square$

5. REMARKS

Finally, we list several remarks.

Remark 5.1. What is the necessary and sufficient condition on $\Omega_{i,j;\ell,m}$ such that the function $\mathfrak{Y}_{i,j;\ell,m;\Omega_{i,j;\ell,m}}(x)$ defined in (1.1) is completely monotonic on $(0, \infty)$?

What is the sharp lower bound of the left hand side inequality in (3.1)?

Remark 5.2. For $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $n > 2$, and two nonnegative integer tuples $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{N}_0^n$ with $\alpha \succ \beta$, let

$$P_{\alpha,\beta;C_{\alpha,\beta}}(x) = \prod_{r=1}^n [(-1)^{\alpha_r} \Phi^{(\alpha_r)}(x)] - C_{\alpha,\beta} \prod_{r=1}^n [(-1)^{\beta_r} \Phi^{(\beta_r)}(x)]$$

on $(0, \infty)$. One can discuss necessary and sufficient conditions on $C_{\alpha,\beta} \in \mathbb{R}$ such that the functions $\pm P_{\alpha,\beta;C_{\alpha,\beta}}(x)$ are respectively completely monotonic on $(0, \infty)$.

Remark 5.3. This paper is the ninth one in a series of articles including [12, 14, 15, 16, 17, 19, 20, 23].

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Email address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com
URL: <https://qifeng618.wordpress.com>, <https://orcid.org/0000-0001-6239-2968>