The affirmative answer to Singer's conjecture on the algebraic transfer of rank four

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ABSTRACT. In recent decades, the structure of the mod-2 cohomology of the Steenrod ring $\mathcal{A}$ has become a major subject of study in the field of Algebraic Topology. One of the earliest attempts to study this cohomology through the use of modular representations of the general linear groups was the groundbreaking work [Math. Z. **202** (1989), 493-523] by W.M. Singer. In that work, Singer introduced a homomorphism, commonly referred to as the "algebraic transfer," which maps from the coinvariants of a certain representation of the general linear group to the mod-2 cohomology group of the ring $\mathcal{A}$. Singer's conjecture, in particular, which states that the algebraic transfer is a monomorphism for all homological degrees, remains a highly significant and unresolved problem in Algebraic Topology. In this research, we take a major stride toward resolving the Singer conjecture by establishing its truth for the homological degree four.

1. Introduction

Everywhere in the text of this article, we will be working over the field $\mathbb{F}_2 \cong \mathbb{Z}/2\mathbb{Z}$ of characteristic 2 and taking (co)homology with coefficients in $\mathbb{F}_2$. It is well-known that the calculation of the stable homotopy groups of spheres $\pi_*^S(\mathbb{S}^0)$ is one of the most central and intractable problems in Algebraic Topology. Historically, in the 1950s, the problem under consideration was approached by Serre [Ser53] through the use of his spectral sequence. Later, in the late 1950s, Adams [Ada58] introduced his celebrated spectral sequence at the prime 2 that converges to $\pi_*^S(\mathbb{S}^0)$. He claimed that E_2 -page of that spectral sequence could be identified with

$$\mathrm{Ext}_{\mathcal{A}}^q(\mathbb{F}_2, \mathbb{F}_2) = \{\mathrm{Ext}_{\mathcal{A}}^{q,t}(\mathbb{F}_2, \mathbb{F}_2) = H^{q,t}(\mathcal{A})\}_{(q,t) \in \mathbb{Z}^2, q \geq 0, t \geq 0},$$

the bigraded cohomology algebra of the classical, singly-graded Steenrod algebra $\mathcal{A}$ over $\mathbb{F}_2$. The explicit computation of this cohomology has been conducted by various researchers: see Adem [Ade52] for $q = 1$, Adams [Ada60] and Wall [Wal60] for $q = 2$, Adams [Ada60] and Wang [Wan67] for $q = 3$, Lin [Lin08] for $q = 4$, Lin [Lin08] and Chen [Che11] for $q = 5$. Despite these developments, the computation of this cohomology remains largely mysterious for any $q > 5$. With the notion that the structure of $\mathrm{Ext}_{\mathcal{A}}^q(\mathbb{F}_2, \mathbb{F}_2)$ can be studied through modular invariant theory, in 1989, W. Singer [Sin89] introduced a transfer homomorphism of rank q that maps from the coinvariants of a certain representation of the general linear group $GL(q)$ over $\mathbb{F}_2$ to the mod-2 cohomology of $\mathcal{A}$. This transfer homomorphism has been the subject of extensive research and has been found to be highly non-trivial. Many studies have been conducted to better understand it, including works by Boardman [Boa91], Minami [Min95], Bruner, Hà, and Hưng [BHH05], Hưng [Hun05], Hà [Ha07], Nam [Nam08], Hưng and Quỳnh [HQ09], Chơn, and Hà [CH11], Sum [Sum18], the author [Phu20, Phu21a, Phu21b, Phu23a, Phu21c, Phu23b, Phu23c, Phu21d, Phu22a, Phu22b], and others. To provide the reader with a comprehensive understanding of this Singer transfer, we will consider the associated objects. Let us denote by $V^{\oplus q} \cong \prod_{1 \leq i \leq q} (\mathbb{Z}/2\mathbb{Z})$, a rank q elementary abelian 2-group which is considered as a q -dimensional $\mathbb{F}_2$ -vector space. It is well-known that $H^*(V^{\oplus q}) \cong S(V_*^{\oplus q})$, the symmetric algebra of the dual space $V_*^{\oplus q} \equiv H^1(V^{\oplus q})$. We can choose $u_1, \dots, u_q$ to be a basis of $H^1(V^{\oplus q})$. In this case, $\mathcal{P}_q := H^*(V^{\oplus q}) \cong \mathbb{F}_2[u_1, \dots, u_q]$, the connected $\mathbb{Z}$ -graded polynomial algebra on generators of degree 1, equipped with the canonical unstable algebra structure over the algebra $\mathcal{A}$. By dualizing, $H_*(V^{\oplus q}) \cong \Gamma(a_1, \dots, a_q)$, the divided power algebra generated by $a_1, \dots, a_q$, each of degree one, where $a_i \equiv a_i^{(1)}$ is dual to u_i . It is to be noted that this algebra and the polynomial algebra $\mathcal{P}_q$ are not in general isomorphic as $\mathbb{F}_2 GL(q)$ -modules. We also knew that the algebra $\mathcal{A}$ is composed of the Steenrod squaring operations Sq^i for $i \geq 0$. The operations Sq^0 and Sq^{2^i} , $i \geq 0$, constitute a system

2010 *Mathematics Subject Classification.* 55Q45, 13A50, 55S10, 55S05, 55T15, 55R12.

Key words and phrases. Adams spectral sequences; primary cohomology operations; Steenrod algebra; lambda algebra; hit problem; invariant theory; algebraic transfer.

This work is dedicated to the people of Ukraine and in memory of the late Professor Reginald Wood.

of multiplicative generators for $\mathcal{A}$ (see also Walker and Wood [WW18]). Emphasizing that these Sq^i are cohomology operations that satisfy the naturality property. Additionally, they commute with the suspension maps and are therefore considered stable. Let $P_{\mathcal{A}}H_*(V^{\oplus q}) \cong \text{Ext}_{\mathcal{A}}^0(\mathcal{P}_q, \mathbb{F}_2)$ be the subspace of $H_*(V^{\oplus q})$ consisting of all elements that are annihilated by all Sq^i for $i > 0$. The group $GL(q)$ acts regularly on $V^{\oplus q}$ and therefore on $\mathcal{P}_q$ and $H_*(V^{\oplus q})$. This action commutes with that of the algebra $\mathcal{A}$ and so acts $\mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_q$ and $P_{\mathcal{A}}H_*(V^{\oplus q})$. Going back to Singer's transfer, in [Sin89], William Singer constructed a homomorphism from $P_{\mathcal{A}}H_n(V^{\oplus q})$ to $\text{Ext}_{\mathcal{A}}^{q, q+n}(\mathbb{F}_2, \mathbb{F}_2)$, which commutes with two Sq^0 's on $P_{\mathcal{A}}H_n(V^{\oplus q})$ and $\text{Ext}_{\mathcal{A}}^{q, q+n}(\mathbb{F}_2, \mathbb{F}_2)$ (see also [Boa91], [Min99]). Further, this map factors through the quotient of its domain's $GL(q)$ -coinvariants to give rise the so-called *algebraic transfer of rank q* ,

$$Tr_q(\mathbb{F}_2) : \mathbb{F}_2 \otimes_{GL(q)} P_{\mathcal{A}}H_n(V^{\oplus q}) \longrightarrow \text{Ext}_{\mathcal{A}}^{q, q+n}(\mathbb{F}_2, \mathbb{F}_2).$$

In actuality, this $Tr_q(\mathbb{F}_2)$ is induced over the E_2 -term of the Adams spectral sequence by the geometrical transfer map, which is a mapping between the suspension spectra in the stable homotopy category, $\Sigma^\infty(B(V^{\oplus q})_+) \longrightarrow \Sigma^\infty(\mathbb{S}^0)$. The results of Singer [Sin89] showed that the "total" transfer $\{Tr_q(\mathbb{F}_2)\}_{q \geq 0}$ is a homomorphism of algebras and that $Tr_q(\mathbb{F}_2)$ is an isomorphism for $q = 1, 2$. This was later expanded upon by Boardman [Boa91], who showed that $Tr_3(\mathbb{F}_2)$ is also an isomorphism. It is noteworthy that during the 1980s, Singer largely held the belief that $Tr_q(\mathbb{F}_2)$ was always an isomorphism for all ranks q . Nevertheless he later claimed that in the rank 5 case, the transfer is not an isomorphism. He demonstrated this by showing that the indecomposable element $Ph_1 \in \text{Ext}_{\mathcal{A}}^{5, 14}(\mathbb{F}_2, \mathbb{F}_2)$ (with P being the Adams periodicity operator) is not in the image of $Tr_5(\mathbb{F}_2)$. This finding tempted him to postulate the following unanswered conjecture.

Conjecture 1.1. ([Sin89, Conj. 1.1]) *$Tr_q(\mathbb{F}_2)$ is a monomorphism for any q .*

According to the above discussion, Singer's transfer is an isomorphism for ranks less than or equal to 3, and thus, Conjecture 1.1 holds true in these ranks. The focus of this paper is the rank 4 case. The aim is to explicitly compute the domain and codomain of the fourth transfer in each "generic" degree n . Using the results in Lin [Lin08], the codomain of $Tr_4(\mathbb{F}_2)$ can be easily determined. Our focus will be on calculating the domain of $Tr_4(\mathbb{F}_2)$, using the known results on the image of $Tr_4(\mathbb{F}_2)$ and the monomial bases of the dual spaces of the primitive spaces $P_{\mathcal{A}}H_n(V^{\oplus 4})$. The challenge in these calculations arises from the fact that $P_{\mathcal{A}}H_n(V^{\oplus 4})$ becomes increasingly larger and more difficult to understand as n increases.

The surjectivity of the transfer map $Tr_q(\mathbb{F}_2)$ is an important aspect of its behavior. As a result, several works, such as [BHH05], [Hum05], [Ha07], [Nam08], and [HQ09], have specifically studied the image of Singer's transfer in the rank 4 case. More precisely, first, Bruner, Hà, and Hưng showed in [BHH05] the following.

Theorem 1.2. ([BHH05, Thm. 1.1]) *For each $s \geq 1$, the non-zero element $g_s \in \text{Ext}_{\mathcal{A}}^{4, 12 \cdot 2^s}(\mathbb{F}_2, \mathbb{F}_2)$ is not in the image of $Tr_4(\mathbb{F}_2)$.*

This result refuted a prediction by Minami in [Min99] that the localization of the algebraic transfer

$$(Sq^0)^{-1}Tr_q(\mathbb{F}_2) : (Sq^0)^{-1}\mathbb{F}_2 \otimes_{GL(q)} P_{\mathcal{A}}H_n(V^{\oplus q}) \longrightarrow (Sq^0)^{-1}\text{Ext}_{\mathcal{A}}^{q, q+n}(\mathbb{F}_2, \mathbb{F}_2)$$

given by inverting Sq^0 is an isomorphism. Specifically, when $q = 4$ and $n = 12 \cdot 2^s - 4$, following [BHH05, Cor. 1.2], the localization of the fourth transfer given by inverting Sq^0 is not an epimorphism. Next, in continuation of the work [BHH05], Hưng further proved in [Hum05] that

Theorem 1.3. ([Hum05, Thm. 1.9]) *Any element in the Sq^0 -families $\{D_3(s) \mid s \geq 0\}$ and $\{p'_s \mid s \geq 0\}$ does not belong to the image of $Tr_4(\mathbb{F}_2)$.*

Owing to Theorems 1.2 and 1.3, there are infinitely many degrees in which the fourth algebraic transfer is not an isomorphism. Further, combining with a technical result (Theorem 1.4) below strengthens a prediction made in Hưng [Hum05, Conj. 1.10].

Theorem 1.4 ([Ha07], [Nam08], [HQ09]). *Every element in the following families belongs to the image of the fourth algebraic transfer $Tr_4(\mathbb{F}_2)$:*

- (i) $\{d_s \mid s \geq 0\}$, $\{e_s \mid s \geq 0\}$ (see Hà [Ha07, Thm. 1.1]);
- (ii) $\{f_s \mid s \geq 0\}$ (see Nam [Nam08, Thm. 1.4]);
- (iii) $\{p_s \mid s \geq 0\}$ (see Hưng and Quỳnh [HQ09, Thm. 1.1]).

Alternatively, since the total transfer $\{Tr_q(\mathbb{F}_2)\}_{q \geq 0}$ is a homomorphism of algebras and $Tr_q(\mathbb{F}_2)$ is an isomorphism for $q = 1, 2, 3$, all decomposable elements in $\text{Ext}_{\mathcal{A}}^4(\mathbb{F}_2, \mathbb{F}_2)$ belong to the image of $Tr_4(\mathbb{F}_2)$.

Combining the results of the studies on the image of $Tr_4(\mathbb{F}_2)$ conducted in [BHH05], [Hum05], [Ha07], [Nam08], and [HQ09], Singer's Conjecture 1.1 for $Tr_4(\mathbb{F}_2)$ can be reformulated as follows.

Conjecture 1.5.

$$\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_n(V^{\oplus 4}) = \begin{cases} \dim \text{Ext}_{\mathcal{A}}^{4,4+n}(\mathbb{F}_2, \mathbb{F}_2) - 1 & \text{if } n \text{ is bad,} \\ \dim \text{Ext}_{\mathcal{A}}^{4,4+n}(\mathbb{F}_2, \mathbb{F}_2) & \text{if } n \text{ is not bad.} \end{cases}$$

Here n is called bad if it equals to the stem of one element in the three families $\{g_s | s \geq 1\}$, $\{D_3(s) | s \geq 0\}$ and $\{p'_s | s \geq 0\}$, whose every element is not in the image of the algebraic transfer of rank four. Otherwise, n is said to be not bad.

Thus, by verifying this conjecture, the answer to Conjecture 1.1 for the fourth transfer can be obtained. This will be discussed in the following section.

The algebraic transfer that is being discussed has a close association with the "hit problem" initiative launched by Frank Peterson [Pet87]. Solving the hit problem is to explicitly determine a minimal generating set for the $\mathbb{F}_2$ -algebra $\mathcal{P}_q$ as an unstable $\mathcal{A}$ -module. The problem has also been motivated by researchers such as Wood [Woo84] and Priddy [Pri90], who have pointed out its connection to modular representation theory, the general linear group, and the stable homotopy type of the classifying space of finite groups. Besides, the close connection of the hit problem with Singer's transfer has led many to believe that the techniques developed in this problem could be utilized to understand the enigmatic structure of Ext groups. The reader is familiar with the fact that if $\mathbb{F}_2$ is an $\mathcal{A}$ -module concentrated in degree 0, then solving the hit problem is equivalent to finding a basis consisting of all equivalence classes of homogeneous polynomials for the $\mathbb{Z}$ -graded vector space over the field $\mathbb{F}_2$:

$$Q\mathcal{P}_q := \mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_q = \{(\mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_q)_n\}_{n \in \mathbb{Z}, n \geq 0} = \mathcal{P}_q / \sum_{i \geq 0} \text{Im}(Sq^{2^i}) = \text{Tor}_0^{\mathcal{A}}(\mathbb{F}_2, \mathcal{P}_q).$$

Where the homogeneous components $(Q\mathcal{P}_q)_n := (\mathbb{F}_2 \otimes_{\mathcal{A}} \mathcal{P}_q)_n$ of degrees n are $\mathbb{F}_2 GL(q)$ -submodules of $Q\mathcal{P}_q$. Usually, people would study the structure of the tensor product $Q\mathcal{P}_q$, which was systematically analyzed by Peterson [Pet87] for $q = 1, 2$, by Kameko's thesis [Kam90] for $q = 3$, and by Sum [Sum14, Sum15] for $q = 4$. The hit problem has received extensive attention from numerous topologists for over three decades, as evidenced by references including [Boa91], [BHH05], [Kam90], [Phu20, Phu21a, Phu21b, Phu23a, Phu23b, Phu22a, Phu22b], [Sum10, Sum14, Sum15, Sum23], [WW18], [Woo89]. Despite the extensive efforts of researchers, the problem remains unsolved for any $q \geq 5$. It should be noted that in general, determining or even approximating the dimension of $Q\mathcal{P}_q$ in each positive degree can be challenging. Kameko put forth a surprising conjecture in his thesis [Kam90] that the order of the factor group $GL(q)/B(q)$ (with $B(q)$ being the Borel subgroup of $GL(q)$) is always an upper bound for the dimension of $(Q\mathcal{P}_q)_n$. This can be phrased as follows:

$$\dim(Q\mathcal{P}_q)_n \leq \text{ord}(GL(q)/B(q)) = \frac{2^{q(q-1)/2} \prod_{1 \leq j \leq q} (2^j - 1)}{2^{q(q-1)/2}} = \prod_{1 \leq j \leq q} (2^j - 1), \text{ for any } n \geq 0.$$

However, this conjecture was later disproven in 2010 by the renowned work of Sum [Sum10]. The well-known μ -function is utilized to simplify the calculation process of $Q\mathcal{P}_q$ in certain positive degrees. The function is defined as follows:

$$\mu : \mathbb{N} \longrightarrow \mathbb{N}, \quad n \longmapsto \min\{k \in \mathbb{N} | n = \sum_{1 \leq i \leq k} (2^{d_i} - 1), d_i > 0, \forall i, 1 \leq i \leq k\}.$$

The following theorem encapsulates the classical results and highlights the importance of this arithmetic function.

Theorem 1.6. *For each non-negative integer n , the following assertions are true:*

- (i) $(Q\mathcal{P}_q)_n = 0$ if and only if $\mu(n) > q$ (see Peterson's conjecture [Pet87], Wood [Woo89]);
- (ii) $(Q\mathcal{P}_q)_n \cong (Q\mathcal{P}_q)_{(n-q)/2}$ if and only if $\mu(n) = q$ (see Kameko's thesis [Kam90]).

The statement (i) is presented by Peterson's conjecture, which was confirmed by Peterson himself for $q \leq 2$. Subsequently, Wood established the general case in a stronger form. Additionally, it is evident that when $\mu(n) > q$, Conjecture 1.1 consistently holds true. It is noteworthy that in [BHP⁺07], the question of whether the codomain of Singer's transfer is trivial if $\mu(n) > q$ was raised. Based on the findings in Adem [Ade52], Adams [Ada60], Wall [Wal60], Wang [Wan67], Lin [Lin08], and Chen [Che11], it has been established that this is the case for $q \leq 5$. Thus, with the aid of Theorem 1.6(i), if this inquiry is confirmed to hold true in the general case, Singer's transfer would be established as a trivial isomorphism. When $\mu(n) = q$, Theorem 1.6(ii) implies that $\mathbb{F}_2 \otimes_{GL(q)} P_{\mathcal{A}} H_n(V^{\oplus q}) \cong \mathbb{F}_2 \otimes_{GL(q)} P_{\mathcal{A}} H_{(n-q)/2}(V^{\oplus q})$. However, this is not necessarily true for the codomains of Singer's transfer in general. For instance, when $n = 16 \cdot 2^s - 6$, we can see that $\mu(n) = 6$ for any $s \geq 2$. Nevertheless, according to Tangora [Tan70] and Bruner [Bru97], $\text{Ext}_{\mathcal{A}}^{6, 16 \cdot 2^3}(\mathbb{F}_2, \mathbb{F}_2) = 0 \neq \text{Ext}_{\mathcal{A}}^{6, 16 \cdot 2^2 = 6 + (16 \cdot 2^3 - 12)/2}(\mathbb{F}_2, \mathbb{F}_2) = \langle D_2 \rangle$, where $D_2 \neq 0$. Taking into account this observation and the fact that the domain of $Tr_q(\mathbb{F}_2)$ is isomorphic to the $GL(q)$ -invariant space $(Q\mathcal{P}_q)_n^{GL(q)}$ as vector spaces for all n , it suffices to investigate Singer's conjecture in internal degrees n for which $\mu(n) \leq q$.

We close this section by recalling the previously established results regarding $\text{Ext}_{\mathcal{A}}^{q,*}(\mathbb{F}_2, \mathbb{F}_2)$ for $q \leq 4$.

Theorem 1.7 ([Ada60], [Ade52], [Wal60], [Wan67], [Lin08]). *The following hold:*

- i) $\text{Ext}_{\mathcal{A}}^{1,*}(\mathbb{F}_2, \mathbb{F}_2)$ is generated by h_i for $i \geq 0$ (see Adem [Ade52]);
- ii) $\text{Ext}_{\mathcal{A}}^{2,*}(\mathbb{F}_2, \mathbb{F}_2)$ is generated by $h_i h_j$ for $j \geq i \geq 0$ and $j \neq i + 1$ (see Adams [Ada60] and Wall [Wal60]);
- iii) $\text{Ext}_{\mathcal{A}}^{3,*}(\mathbb{F}_2, \mathbb{F}_2)$ is generated by $h_i h_j h_\ell$, c_t for $t \geq 0$; $\ell \geq j \geq i \geq 0$, and subject only to the relations $h_i h_{i+1} = 0$, $h_i h_{i+2}^2 = 0$ and $h_i^3 = h_{i-1}^2 h_{i+1}$ (see Adams [Ada60] and Wang [Wan67]);
- iv) $\text{Ext}_{\mathcal{A}}^{4,*}(\mathbb{F}_2, \mathbb{F}_2)$ is generated by $h_i h_j h_\ell h_m$, $h_u c_v$, d_t , e_t , f_t , g_{t+1} , p_t , $D_3(t)$, p'_t for $m \geq \ell \geq j \geq i \geq 0$, $u, v, t \geq 0$, and subject to the relations in iii) together with $h_i^2 h_{i+3}^2 = 0$, $h_{v-1} c_v = 0$, $h_v c_v = 0$, $h_{v+2} c_v = 0$ and $h_{v+3} c_v = 0$ (see Lin [Lin08]).

2. A solution to Singer's conjecture on the rank 4 transfer

The purpose of this section, as outlined in the introduction, is to validate Singer's conjecture regarding the fourth transfer homomorphism. This will be achieved by proving that Conjecture 1.5 is valid for internal degrees n where $\mu(n) \leq 4$.

• **Consider $\mu(n) < 4$.** According to Sum [Sum14], the degree n takes on one of the following "generic" forms:

$$(2.1) \quad \begin{aligned} i) \quad n &= 2^{s+1} - t, \quad t \in \{1, 2, 3\}; \\ ii) \quad n &= 2^{s+t+1} + 2^{s+1} - 3; \\ iii) \quad n &= 2^{s+t} + 2^s - 2; \\ iv) \quad n &= 2^{s+t+u} + 2^{s+t} + 2^s - 3, \end{aligned}$$

whenever s, t, u are the positive integers. Then, we establish the following technical outcome.

Main Theorem. *Let us consider generic degrees specified in equation (2.1). Consequently, Conjecture 1.5 holds true in these degrees. Additionally, the transfer homomorphism $Tr_4(\mathbb{F}_2)$ is an isomorphism, except for the cases specified in item i) with $(s, t) = (6, 2), (5, 3)$, item ii) with $(s, t) = (2, 3)$, and item iii) with $(s, t) = (4, 3), (1, 7)$. In these cases, $Tr_4(\mathbb{F}_2)$ is only a monomorphism and not an epimorphism.*

The theorem has been proven by Bruner, Hà, and Hưng [BHH05, Prop. 4.4] for item iii) with $(s, t) = (1, 2)$, by Hưng [Hun05, Thm. 7.3] for item iii) with $(s, t) \in \{(1, 7), (4, 3)\}$, by Sum [Sum18, Thm. 4.1] for item i), and by the present author in [Phu21c, Phu23b] for items ii), iii) where $t \neq 3$ and in [Phu21d] item iv). It is remarkable that in [Phu21c, Phu23b], the theorem was proven for the cases $(s, t) \in \{(1, 2), (1, 7)\}$ using another method. Hence, it remains to prove the theorem for items ii) and iii) with $t = 3$. Although the case $(s, t) = (4, 3)$ has already been proven by Hưng [Hun05], it will be redemonstrated in this paper using other techniques. Before delving into the details, the known results will be briefly presented for the reader's benefit.

We will begin by discussing the theorem for item i), which has been established by Sum in [Sum18].

Case $n := n_{s,t} = 2^{s+1} - t$, $1 \leq t \leq 3$, $s \geq 1$. According to Sum [Sum18], the dimension of the domain of $Tr_4(\mathbb{F}_2)$ in degree $n_{s,t}$ is determined by

$$(2.2) \quad \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{s,t}}(V^{\oplus 4}) = \begin{cases} 0 & \text{if } t = 1, \text{ and } s = 1, \\ 1 & \text{if } t = 1, \text{ and } s \geq 2, \\ 0 & \text{if } t = 2, \text{ and } 1 \leq s \leq 2, \\ 1 & \text{if } t = 2, \text{ and } s \geq 3, \\ 0 & \text{if } t = 3, \text{ and } s \geq 1. \end{cases}$$

On the other side, by Theorem 1.7, one gets

$$(2.3) \quad \text{Ext}_{\mathcal{A}}^{4,4+n_{s,t}}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} 0 & \text{if } t = 1, \text{ and } s = 1, \\ \langle h_0^3 h_{s+1} \rangle & \text{if } t = 1, \text{ and } s \geq 2, \\ 0 & \text{if } t = 2, \text{ and } 1 \leq s \leq 2, \\ \langle d_0 \rangle & \text{if } t = 2, \text{ and } s = 3, \\ \langle h_0^2 h_s^2 \rangle & \text{if } t = 2, \text{ and } s \geq 4, s \neq 6, \\ \langle h_0^2 h_6^2, D_3(1) \rangle & \text{if } t = 2, \text{ and } s = 6, \\ \langle D_3(0) \rangle & \text{if } t = 3, \text{ and } s = 5, \\ 0 & \text{if } t = 3, \text{ and } s \geq 6. \end{cases}$$

We see that $n_{s,t}$ is bad for $(s, t) = (6, 2), (5, 3)$ and that by the equalities (2.2), (2.3), Conjecture 1.5 holds for the degrees $n_{s,t}$ for $1 \leq t \leq 3$ and arbitrary positive integer s .

Notice that the non-zero elements $h_0^3 h_{s+1}$, $h_0^2 h_s^2$, and $h_0^2 h_6^2$ are decomposable in the fourth cohomology groups $\text{Ext}_{\mathcal{A}}^{4,4+n_{s,t}}(\mathbb{F}_2, \mathbb{F}_2)$, and so they are in the image of $Tr_4(\mathbb{F}_2)$. It is well-known that by Hưng [Hun05], the indecomposable elements $D_3(0)$ and $D_3(1)$ are not in the image of $Tr_4(\mathbb{F}_2)$ (see also Theorem 1.3) and that by Hà [Ha07], the indecomposable element d_0 is in the image of $Tr_4(\mathbb{F}_2)$ (see also Theorem 1.4(i)). These results together with the equalities (2.2) and (2.3) show that $Tr_4(\mathbb{F}_2)$ is an isomorphism in degrees $n_{s,t}$, except the degrees $n_{6,2}$ and $n_{5,3}$. In degrees $n_{6,2}$ and $n_{5,3}$, the fourth transfer is a monomorphism, but it fails to be an epimorphism. This is due to the fact that the dimension of $\mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{6,2}}(V^{\oplus 4})$ is 1, which is strictly less than the dimension of $\text{Ext}_{\mathcal{A}}^{4,4+n_{6,2}}(\mathbb{F}_2, \mathbb{F}_2)$, which is 2. Similarly, the dimension of $\mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{5,3}}(V^{\oplus 4})$ is 0, which is strictly less than the dimension of $\text{Ext}_{\mathcal{A}}^{4,4+n_{5,3}}(\mathbb{F}_2, \mathbb{F}_2)$, which is 1.

Next, we will examine the theorem for items ii), iii) where $t \neq 3$, and item iv). In item iii), the result has also been established by Bruner, Hà, and Hưng in [BHH05] for the cases $(s, t) = (1, 2)$ and by Hưng in [Hun05] for the case $(s, t) = (1, 7)$. The others cases in item iii) have been established by the author in [Phu21c, Phu23b, Phu21d]. More precisely, with regards to item ii), we are concerned with the following case.

Case $n := n_{s,t} = 2^{s+t+1} + 2^{s+1} - 3$, $t \geq 1$, $t \neq 3$, $s \geq 1$. Our previous works [Phu21c, Phu23b] have shown that

$$(2.4) \quad \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{s,t}}(V^{\oplus 4}) = \begin{cases} 1 & \text{if } t = 1, \text{ and } s = 1, \\ 1 & \text{if } t = 1, \text{ and } s \geq 2, \\ 1 & \text{if } t = 2, \text{ and } s = 1, \\ 0 & \text{if } t = 2, \text{ and } s \geq 2, \\ 1 & \text{if } t \geq 4, \text{ and } 1 \leq s \leq 2, \\ 2 & \text{if } t \geq 4, \text{ and } s \geq 3. \end{cases}$$

On the other hand, from Theorem 1.7, we get

$$(2.5) \quad \text{Ext}_{\mathcal{A}}^{4,4+n_{s,t}}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle h_1 c_0 \rangle & \text{if } t = 1, \text{ and } s = 1, \\ \langle h_0 h_{s+1}^3 \rangle & \text{if } t = 1, \text{ and } s \geq 2, \\ \langle e_0 \rangle & \text{if } t = 2, \text{ and } s = 1, \\ 0 & \text{if } t = 2, \text{ and } s \geq 2, \\ \langle h_0 h_{s+1} h_{s+t}^2 \rangle & \text{if } t \geq 4, \text{ and } 1 \leq s \leq 2, \\ \langle h_0 h_s^2 h_{s+t+1}, h_0 h_{s+1} h_{s+t}^2 \rangle & \text{if } t \geq 4, \text{ and } s \geq 3. \end{cases}$$

Then, the equalities (2.4) and (2.5) show that $n_{s,t}$ is not bad and Conjecture 1.5 also holds for the degree $n_{s,t}$ whenever $t \geq 1$, $t \neq 3$, and $s \geq 1$.

As is well known, $\{h_j | j \geq 0\} \subset \text{Im}(Tr_1(\mathbb{F}_2))$ (see Singer [Sin89]), $\{c_j | j \geq 0\} \subset \text{Im}(Tr_3(\mathbb{F}_2))$ (see Boardman [Boa91]), $\{e_j | j \geq 0\} \subset \text{Im}(Tr_4(\mathbb{F}_2))$ (see Hà [Ha07] and Theorem 1.4(i)), and the "total" transfer $\{Tr_q(\mathbb{F}_2)\}_{q \geq 0}$ is an algebra homomorphism (see Singer [Sin89]). So, together with the equalities (2.4) and (2.5), we assert that $Tr_4(\mathbb{F}_2)$ is an isomorphism in degrees $n_{s,t}$ for each $t \geq 1$, $t \neq 3$ and all $s \geq 1$.

Next, for item iii) with $t \neq 3$, the following situation occurs.

Case $n := n_{s,t} = 2^{s+t} + 2^s - 2$, $t \geq 1$, $t \neq 3$, $s \geq 1$. By Theorem 1.7, we obtain

$$(2.6) \quad \text{Ext}_{\mathcal{A}}^{4,4+n_{s,t}}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} 0 & \text{if } t = 1, \text{ and } 1 \leq s \leq 4, s \neq 3, \\ \langle h_2 c_1 \rangle & \text{if } t = 1, \text{ and } s = 3, \\ \langle h_1 h_s^3 \rangle & \text{if } t = 1, \text{ and } s \geq 5, \\ 0 & \text{if } t = 2, \text{ and } s = 1, \\ \langle h_0^2 h_2 h_4, f_0 \rangle & \text{if } t = 2, \text{ and } s = 2, \\ \langle h_0^2 h_3 h_5, e_1 \rangle & \text{if } t = 2, \text{ and } s = 3, \\ \langle h_0^2 h_s h_{s+2} \rangle & \text{if } t = 2, \text{ and } s \geq 4, \\ \langle d_1 \rangle & \text{if } t = 4, \text{ and } s = 1, \\ \langle h_1^3 h_6 \rangle & \text{if } t = 4, \text{ and } s = 2, \\ \langle h_0^2 h_3 h_7, h_1 h_3 h_6^2 \rangle & \text{if } t = 4, \text{ and } s = 3, \\ \langle h_1 h_4 h_7^3 \rangle & \text{if } t = 4, \text{ and } s = 4, \\ \langle h_1 h_{s-1}^2 h_{s+4}, h_1 h_s h_{s+3}^2 \rangle & \text{if } t = 4, \text{ and } s \geq 5, \\ \langle h_1^2 h_t^2 \rangle & \text{if } t \geq 5, t \neq 7, \text{ and } s = 1, \\ \langle h_1^2 h_7^2, D_3(2) \rangle & \text{if } t = 7, \text{ and } s = 1, \\ \langle h_1^3 h_{t+2} \rangle & \text{if } t \geq 5, \text{ and } s = 2, \\ \langle h_1 h_3 h_{t+2}^2, h_0^2 h_3 h_{t+3} \rangle & \text{if } t \geq 5, \text{ and } s = 3, \\ \langle h_1 h_3 h_{t+3}^2, h_0^2 h_4 h_{t+4} \rangle & \text{if } t \geq 5, \text{ and } s = 4, \\ \langle h_1 h_s h_{s+t-1}^2, h_0^2 h_s h_{s+t}, h_1 h_{s-1}^2 h_{s+t} \rangle & \text{if } t \geq 5, \text{ and } s \geq 5. \end{cases}$$

Based on the results in Bruner, Hà, and Hưng [BHH05, Prop. 4.4], Hưng [Hun05, Thm. 7.3] and our previous papers [Phu21c, Phu23b], the dimension of the domain of $Tr_4(\mathbb{F}_2)$ in degree $n_{s,t}$ is determined by

$$(2.7) \quad \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{s,t}}(V^{\oplus 4}) = \begin{cases} 0 & \text{if } t = 1, \text{ and } 1 \leq s \leq 4, s \neq 3, \\ 1 & \text{if } t = 1, \text{ and } s \geq 3, s \neq 4, \\ 0 & \text{if } t = 2, \text{ and } s = 1 \text{ (see [BHH05, Prop. 4.4])}, \\ 2 & \text{if } t = 2, \text{ and } 2 \leq s \leq 3, \\ 1 & \text{if } t = 2, \text{ and } s \geq 4, \\ 1 & \text{if } t = 4, \text{ and } 1 \leq s \leq 4, s \neq 3, \\ 2 & \text{if } t = 4, \text{ and } s \geq 3, s \neq 4, \\ 1 & \text{if } t = 7, \text{ and } s = 1 \text{ (see [Hun05, Thm. 7.3])}, \\ 1 & \text{if } t \geq 5, t \neq 7, \text{ and } s = 1, \\ 1 & \text{if } t \geq 5, \text{ and } s = 2, \\ 2 & \text{if } t \geq 5, \text{ and } 3 \leq s \leq 4, \\ 3 & \text{if } t \geq 5, \text{ and } s \geq 5. \end{cases}$$

From the equations (2.6) and (2.7), it can be deduced that the only degree $n_{1,7}$ is considered bad, and the Conjecture 1.5 holds true for the degree $n_{s,t}$ whenever $t \geq 1$, $t \neq 3$ and $s \geq 1$.

Based on Theorems 1.3 and 1.4, the equalities (2.6) and (2.7), and the fact that $\{h_s\}_{s \geq 0}$ is a subset of $\text{Im}(Tr_1(\mathbb{F}_2))$ and the total algebraic transfer $\{Tr_q(\mathbb{F}_2)\}_{q \geq 0}$ forms a homomorphism of algebras (as stated in Singer [Sin89]), it can be concluded that $Tr_4(\mathbb{F}_2)$ is an isomorphism in degrees $n_{s,t}$ for all $t \geq 1$ with $t \notin 3, 7$ and for all $s \geq 1$. According to Hưng [Hun05, Thm. 7.3], when $s = 1$ and $t = 7$, the fourth transfer is proven to be a monomorphism, yet it fails to be an epimorphism in the internal degree $n_{1,7}$.

Case $n := n_{s,u,t} = 2^{s+t+u} + 2^{s+t} + 2^s - 3$, $s \geq 1$, $u \geq 1$, $t \geq 1$. By Theorem 1.7 and the results presented in our previous work [Phu21d], the following conclusions can be drawn:

$$(2.8) \quad \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_s, u, t}(V^{\oplus 4}) = \begin{cases} 1 & \text{if } s = 1, u = 2, \text{ and } t \geq 2, \\ 1 & \text{if } s = 2, u \geq 1, \text{ and } t = 1, \\ 1 & \text{if } s \geq 2, u \geq 2, \text{ and } t \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

$$(2.9) \quad \text{Ext}_{\mathcal{A}}^{4, 4+n_s, u, t}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle h_0 c_t \rangle & \text{if } s = 1, u = 2, \text{ and } t \geq 2, \\ \langle h_{u+3} c_0 \rangle & \text{if } s = 2, u \geq 1, \text{ and } t = 1, \\ \langle h_0 h_t h_{s+t} h_{s+t+u} \rangle & \text{if } s \geq 2, u \geq 2, \text{ and } t \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

We must take note of the remaining cases for s, u, t where

$$\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_s, u, t}(V^{\oplus 4}) = 0 = \dim \text{Ext}_{\mathcal{A}}^{4, 4+n_s, u, t}(\mathbb{F}_2, \mathbb{F}_2).$$

These cases will be described as follows: $s = 1, u = 1$ and $t \geq 1$; $s = 2, u = 1$ and $t \geq 2$; $s \geq 3, u = 1$ and $t \geq 1$; $s = 1, u = 2$ and $t = 1$; $s = 1, u \geq 3$ and $t \geq 1$; $s \geq 3, u \geq 2$ and $t = 1$. It is straightforward to see that $n_{s, u, t}$ is not bad for these circumstances, and that by the equalities (2.8), (2.9), Conjecture 1.5 holds in the degree $n_{s, u, t}$ for every $s \geq 1, u \neq 1$ and $t \geq 1$.

Because $\{h_s | s \geq 0\} \subset \text{Im}(Tr_1(\mathbb{F}_2))$ (see Singer [Sin89]) and $\{c_s | s \geq 0\} \subset \text{Im}(Tr_3(\mathbb{F}_2))$ (see Boardman [Boa91]) and the total transfer $\{Tr_q(\mathbb{F}_2)\}_{q \geq 0}$ is a homomorphism of algebras, by equalities (2.8) and (2.9), we may assert that $Tr_4(\mathbb{F}_2)$ is an isomorphism in the degrees $n_{s, u, t}$ whenever $s \geq 1, u \geq 1, t \geq 1$.

Following [Sin89] and [Boa91], we have $\{h_s\}_{s \geq 0} \subset Tr_1(\mathbb{F}_2)$ and $\{c_s\}_{s \geq 0} \subset Tr_3(\mathbb{F}_2)$. Combining these together with equations (2.8) and (2.9) and the fact that the total transfer $\{Tr_q(\mathbb{F}_2)\}_{q \geq 0}$ is a homomorphism of algebras show that $Tr_4(\mathbb{F}_2)$ is an isomorphism in degrees $n_{s, u, t}$ for all $s \geq 1, u \geq 1, t \geq 1$.

We will now proceed to demonstrate the Main Theorem for items ii) and iii) in the case where $t = 3$. It is crucial to acknowledge that the calculation method utilized in this study is similar to that employed in our previous works in [Phu21c, Phu23b, Phu21d].

Proof of Main Theorem. We will commence by investigating item ii) with $t = 3$, which corresponds to degree $n := n_{s, 3} = 2^{s+4} + 2^{s+1} - 3$ for all $s \geq 1$. It is important to remember that the dimension of $(QP_4)_{n_{s, 3}}$ has been established by Sum in [Sum14] as follows:

$n_{s, 3}$	$s = 1$	$s = 2$	$s \geq 3$
$\dim(QP_4)_{n_{s, 3}}$	136	180	195

A monomial basis of $(QP_4)_{n_{s, 3}}$ is provided in [Sum14]. By utilizing this basis and a computational method similar to those employed in our previous works [Phu23b, Phu21d], we deduce that the $GL(4)$ -invariant subspace $(QP_4)_{n_{s, 3}}^{GL(4)}$ is null if $s = 2$ and has a dimension of 1 if $s \neq 2$. As it is known, $\mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_s, 3}(V^{\oplus 4}) \cong (QP_4)_{n_{s, 3}}^{GL(4)}$, the dimensions of the domain of $Tr_4(\mathbb{F}_2)$ in degrees $n_{s, 3}$ are determined by

$$(2.10) \quad \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_s, 3}(V^{\oplus 4}) = \begin{cases} 0 & \text{if } s = 2, \\ 1 & \text{otherwise.} \end{cases}$$

On the other hand, it follows from Lin's result [Lin08] that

$$(2.11) \quad \text{Ext}_{\mathcal{A}}^{4, 4+n_s, 3}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle p_0 \rangle & \text{if } s = 1, \\ \langle p'_0 \rangle & \text{if } s = 2, \\ \langle h_0 h_{s+1} h_{s+3}^2, h_0 h_s^2 h_{s+4} \rangle = \langle h_0 h_s^2 h_{s+4} \rangle & \text{if } s \geq 3. \end{cases}$$

Hence, in view of the equalities (2.10) and (2.11), it can be deduced that the degree $n_{s, 3}$ is bad for $s = 2$, and that Conjecture 1.5 holds true for the degrees $n_{s, 3} = 2^{s+4} + 2^{s+1} - 3$ for all integers $s \geq 1$. Further the fourth transfer

$$Tr_4(\mathbb{F}_2) : \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_s, 3}(V^{\oplus 4}) \longrightarrow \text{Ext}_{\mathcal{A}}^{4, 4+n_s, 3}(\mathbb{F}_2, \mathbb{F}_2)$$

is an isomorphism for $s \neq 2$, and a monomorphism, but not an epimorphism, for $s = 2$. This conclusion is based on the fact that $h_0 h_s^2 h_{s+4} \in \text{Im}(Tr_4(\mathbb{F}_2))$, Theorems 1.3 and 1.4, and the equalities (2.10) and (2.11).

Finally, we consider item iii) with $t = 3$, which corresponds to degree $n_{s,3} = 2^{s+3} + 2^s - 2$ for all $s \geq 1$. Notice that $n_{s,3}$ is an even degree, and so, the Kameko map

$$(\overline{Sq}^0)_{n_{s,3}} := \overline{Sq}^0 : (Q\mathcal{P}_4)_{n_{s,3}} \rightarrow (Q\mathcal{P}_4)_{(n_{s,3}-4)/2} \equiv (Q\mathcal{P}_4)_{2^{s+2}+2^{s-1}-3}$$

is an epimorphism of $\mathbb{F}_2 GL(4)$ -modules. This leads to an estimate

$$(2.12) \quad \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{s,3}}(V^{\oplus 4}) \leq \dim[(\text{Ker}((\overline{Sq}^0)_{n_{s,3}}))^{GL(4)}]^* + \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{2^{s+2}+2^{s-1}-3}(V^{\oplus 4}).$$

Here $[(\text{Ker}((\overline{Sq}^0)_{n_{s,3}}))^{GL(4)}]^*$ denotes the dual of $(\text{Ker}((\overline{Sq}^0)_{n_{s,3}}))^{GL(4)}$. Following Sum [Sum18] and the equality (2.10), the coinvariant $\mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{2^{s+2}+2^{s-1}-3}(V^{\oplus 4})$ is trivial if $s = 1, 4$ and is 1-dimensional otherwise. We need to compute the dimension of the $GL(4)$ -invariant $(\text{Ker}((\overline{Sq}^0)_{n_{s,3}}))^{GL(4)}$. By Sum [Sum14, Sum15], the dimension of the kernel of $(\overline{Sq}^0)_{n_{s,3}}$ is determined as follows:

$n_{s,3}$	$s = 1$	$s = 2$	$s \geq 3$
$\dim \text{Ker}((\overline{Sq}^0)_{n_{s,3}})$	49	90	105

By utilizing this result and performing a calculation similar to those in [Phu21c, Phu21d], we assert that the invariant space $(\text{Ker}((\overline{Sq}^0)_{n_{s,3}}))^{GL(4)}$ is trivial when $s = 1, 2$ and has a dimension of 1 when $s \geq 3$. Based on these findings, in conjunction with inequality (2.12), we conclude that

$$(2.13) \quad 0 \leq \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{s,3}}(V^{\oplus 4}) \leq \begin{cases} 0 & \text{if } s = 1, \\ 1 & \text{if } s = 2, 4, \\ 2 & \text{otherwise.} \end{cases}$$

On the other hand, as a result of Lin's work [Lin08], it has been discovered that

$$(2.14) \quad \text{Ext}_{\mathcal{A}}^{4,4+n_{s,3}}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle h_1^2 h_3^2 \rangle = 0 & \text{if } s = 1, \\ \langle h_0^2 h_2 h_5, h_1 h_2 h_4^2 \rangle = \langle h_1^3 h_5 \rangle & \text{if } s = 2, \\ \langle h_0^2 h_3 h_6, p_1 \rangle & \text{if } s = 3, \\ \langle h_0^2 h_4 h_7, p_1' \rangle & \text{if } s = 4, \\ \langle h_0^2 h_s h_{s+3}, h_1 h_{s-1}^2 h_{s+3} \rangle & \text{if } s \geq 5. \end{cases}$$

It has been established that the non-zero elements $h_1^3 h_5$, $h_0^2 h_s h_{s+3}$ for $s \geq 3$, and $h_1 h_{s-1}^2 h_{s+3}$ for $s \geq 5$ are detected by the fourth transfer. In reality, this can be directly proved, following the methods used in our previous works [Phu21c, Phu23b, Phu21d], by utilizing the E_1 -level of $Tr_4(\mathbb{F}_2)$. For example, a straightforward computation shows that

$$\begin{aligned} Tr_4(\mathbb{F}_2)([a_1^{(1)} a_2^{(2^{s-1}-1)} a_3^{(2^{s-1}-1)} a_4^{(2^{s+3}-1)}]) &= [\psi_4(a_1^{(1)} a_2^{(2^{s-1}-1)} a_3^{(2^{s-1}-1)} a_4^{(2^{s+3}-1)})] \\ &= [\lambda_1 \lambda_{2^{s-1}-1}^2 \lambda_{2^{s+3}-1}] = h_1 h_{s-1}^2 h_{s+3}, \end{aligned}$$

where $a_1^{(1)} a_2^{(2^{s-1}-1)} a_3^{(2^{s-1}-1)} a_4^{(2^{s+3}-1)} \in P_{\mathcal{A}} H_{n_{s,3}}(V^{\oplus 4})$, and the linear transformation ψ_4 is viewed as a representation in the lambda algebra of $Tr_4(\mathbb{F}_2)$ and determined by the inductive formula as described in [CH11]. The above equality implies that $h_1 h_{s-1}^2 h_{s+3} \in \text{Im}(Tr_4(\mathbb{F}_2))$ for every $s \geq 5$. Hence, combining (2.14) and Theorems 1.3 and 1.4 gives

$$(2.15) \quad \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{s,3}}(V^{\oplus 4}) \geq \begin{cases} 0 & \text{if } s = 1, \\ 1 & \text{if } s = 2, 4, \\ 2 & \text{otherwise.} \end{cases}$$

Noting that thanks to (2.13) and (2.15), the coinvariant space $\mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{s,3}}(V^{\oplus 4})$ is 1-dimensional through. This result was also achieved by Hưng [Hun05, Thm. 7.3] through computer calculations. So, Conjecture 1.5 holds for the degrees $n_{4,3} = 2^{4+3} + 2^4 - 2$. It is easy to see that the only degree $n_{4,3}$ is bad. This data, combined with the inequalities in (2.13), (2.14), and (2.15), suggests that Conjecture 1.5 holds true for the degrees $n_{s,3} = 2^{s+3} + 2^s - 2$ for all positive integers s .

It is well-known that by Hưng [Hun05], the indecomposable element p_1' is not in the image of $Tr_4(\mathbb{F}_2)$ (see also Theorem 1.3) and that by Hưng and Quỳnh [HQ09], the indecomposable element p_1 is in the image of $Tr_4(\mathbb{F}_2)$ (see also Theorem 1.4(iii)). At the same time, as shown above, the decomposable elements $h_1^3 h_5$, $h_0^2 h_s h_{s+3}$, for $s \geq 3$, and $h_1 h_{s-1}^2 h_{s+3}$ for $s \geq 5$ are in the image of $Tr_4(\mathbb{F}_2)$. Therefore,

by combining the inequalities expressed in (2.13), (2.14), and (2.15), we can conclude that the fourth transfer,

$$Tr_4(\mathbb{F}_2) : \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_s,3}(V^{\oplus 4}) \longrightarrow \text{Ext}_{\mathcal{A}}^{4,4+n_s,3}(\mathbb{F}_2, \mathbb{F}_2)$$

is an isomorphism for $s \neq 4$, and a monomorphism but not an epimorphism for $s = 4$. The proof of the theorem is complete. $\square$

• **Consider $\mu(n) = 4$.** By Theorem 1.6(ii), this case is an immediate consequence of the results in [Sin89], [BHH05], [Hum05], Theorems 1.2, 1.3, 1.4 and Main Theorem. Indeed, we obtain the following corollaries.

Corollary 2.1. *Let us consider the following generic degrees:*

$$\begin{aligned} n &:= n_s^{(1)} = 2^{s+4} + 2^{s+1} - 4, \\ n &:= n_s^{(2)} = 2^{s+4} + 2^{s+2} + 2^s - 4, \\ n &:= n_s^{(3)} = 2^{s+4} + 2^{s+2} + 2^{s+1} - 4, \\ n &:= n_s^{(4)} = 2^{s+5} + 2^{s+2} + 2^s - 4, \end{aligned}$$

where s is an arbitrary positive integer. Then, Conjecture 1.5 holds true in these degrees and Singer's transfer is an isomorphism in the bidegree $(4, 4 + n_s^{(j)})$ for $1 \leq j \leq 4$ and all $s > 0$.

It is of significance to bear in mind that, in every degree $n_s^{(j)}$, the case where $s = 0$ is excluded from consideration, as it has already been thoroughly analyzed in the proof of the Main Theorem. Remark that by the previous works in Hà [Ha07], Nam [Nam08], Hưng and Quỳnh [HQ09], Singer's transfer is an epimorphism in the bidegree $(4, 4 + n_s^{(j)})$ for $1 \leq j \leq 4$ and all $s > 0$. Indeed, following Theorem 1.7, for each positive integer s , we have

$$(2.15) \quad \text{Ext}_{\mathcal{A}}^{4,4+n_s^{(j)}}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle d_s \rangle & \text{if } j = 1, \\ \langle h_{s-1}^2 h_{s+2} h_{s+4}, e_s \rangle & \text{if } j = 2, \\ \langle h_{s-1}^2 h_{s+1} h_{s+3} = h_s^3 h_{s+3}, f_s \rangle & \text{if } j = 3, \\ \langle h_{s-1}^2 h_{s+2} h_{s+5}, p_s \rangle & \text{if } j = 4. \end{cases}$$

By Singer [Sin89], the decomposable elements $h_{s-1}^2 h_{s+2} h_{s+4}$, $h_{s-1}^2 h_{s+1} h_{s+3}$ and $h_{s-1}^2 h_{s+2} h_{s+5}$ are detected by the fourth transfer, $Tr_4(\mathbb{F}_2)$ for any $s > 0$. On the other hand, by Theorem 1.4, $\{d_s | s \geq 0\} \subset \text{Im}(Tr_4(\mathbb{F}_2))$ (see Hà [Ha07]), $\{e_s | s \geq 0\} \subset \text{Im}(Tr_4(\mathbb{F}_2))$ (see Hà [Ha07]), $\{f_s | s \geq 0\} \subset \text{Im}(Tr_4(\mathbb{F}_2))$ (see Nam [Nam08]) and $\{p_s | s \geq 0\} \subset \text{Im}(Tr_4(\mathbb{F}_2))$ (see Hưng and Quỳnh [HQ09]). So, the degrees $n_s^{(j)}$ are not bad and $Tr_4(\mathbb{F}_2)$ is an epimorphism in those degrees for $1 \leq j \leq 4$ and any $s > 0$. Thus, to firmly establish that $Tr_4(\mathbb{F}_2)$ is an isomorphism in degree $n_s^{(j)}$, it is imperative to prove Conjecture 1.5 to be true in this degree. The proof will be structured in the following manner.

Proof. It is straightforward to check that $\mu(n_s^{(3)}) = 4$ for every $s > 0$ and $\mu(n_s^{(j)}) = 4$ for every $s > 1$ and $j \neq 3$. Then, combining with Theorem 1.6(ii), one has the following isomorphisms:

$$(2.16) \quad \begin{aligned} \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_s^{(3)}}(V^{\oplus 4}) &\cong \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_0^{(3)}}(V^{\oplus 4}), \quad \text{for all } s \geq 1, \\ \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_s^{(j)}}(V^{\oplus 4}) &\cong \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_1^{(j)}}(V^{\oplus 4}), \quad \text{for all } s \geq 2 \text{ and } j \neq 3. \end{aligned}$$

Due to the equality (2.7), and the inequalities (2.13), (2.15) in the proof of Main Theorem, we get

$$(2.17) \quad \begin{aligned} \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_0^{(3)}}(V^{\oplus 4}) &= \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{2,2}}(V^{\oplus 4}) = 2, \\ \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_1^{(j)}}(V^{\oplus 4}) &= \begin{cases} \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{1,4}}(V^{\oplus 4}) = 1 & \text{if } j = 1, \\ \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{3,2}}(V^{\oplus 4}) = 2 & \text{if } j = 2, \\ \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{3,3}}(V^{\oplus 4}) = 2 & \text{if } j = 4. \end{cases} \end{aligned}$$

Indeed, with $n_0^{(3)} = 2^{0+4} + 2^{0+2} + 2^{0+1} - 4 = 18$, according to (2.7), $\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{2,2}}(V^{\oplus 4}) = 2$, where $n_{2,2} = 2^{2+2} + 2^2 - 2 = 18$. So, $n_0^{(3)} \equiv n_{2,2} = 18$ and

$$\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_0^{(3)}}(V^{\oplus 4}) = \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{2,2}}(V^{\oplus 4}) = 2.$$

Next, one has $n_1^{(1)} = 2^{1+4} + 2^{1+1} - 4 = 32$. Then, due to (2.7), $\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{1,4}}(V^{\oplus 4}) = 1$, where $n_{1,4} = 2^{1+4} + 2^1 - 2 = 32$. So, $n_1^{(1)} \equiv n_{1,4} = 32$ and

$$\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_1^{(1)}}(V^{\oplus 4}) = \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{1,4}}(V^{\oplus 4}) = 1.$$

Next, notice that $n_1^{(2)} = 2^{1+4} + 2^{1+2} + 2^1 - 4 = 38$, and following (2.7), $\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{3,2}}(V^{\oplus 4}) = 2$, where $n_{3,2} = 2^{3+2} + 2^3 - 2 = 38$. So, $n_1^{(2)} \equiv n_{3,2} = 38$ and

$$\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_1^{(2)}}(V^{\oplus 4}) = \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{3,2}}(V^{\oplus 4}) = 2.$$

Finally, with $n_1^{(4)} = 2^{1+5} + 2^{1+2} + 2^1 - 4 = 70$, by (2.13) and (2.15), $\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{3,3}}(V^{\oplus 4}) = 2$, where $n_{3,3} = 2^{3+3} + 2^3 - 2 = 70$. So, $n_1^{(4)} \equiv n_{3,3} = 70$ and

$$\dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_1^{(4)}}(V^{\oplus 4}) = \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_{3,3}}(V^{\oplus 4}) = 2.$$

Thus, the equalities (2.15), (2.16) and (2.17) indicated that Conjecture 1.5 is true in degrees $n_s^{(j)}$ with $1 \leq j \leq 4$ and s an arbitrary positive integer. The corollary follows. $\square$

Corollary 2.2. *Let us consider the following generic degrees:*

$$\begin{aligned} n &:= n_s^{(5)} = 2^{s+3} + 2^{s+2} - 4, \\ n &:= n_s^{(6)} = 2^{s+6} + 2^{s+3} + 2^s - 4, \\ n &:= n_s^{(7)} = 2^{s+6} + 2^s - 4, \end{aligned}$$

where s is an arbitrary positive integer. Then, Conjecture 1.5 also holds true in these degrees and Singer's transfer is a monomorphism, but it is not an epimorphism in the bidegree $(4, 4 + n_s^{(j)})$ for $5 \leq j \leq 7$ and arbitrary $s \geq 1$.

We remark that by the previous works in [BHH05] and [Hun05], Singer's transfer is not an epimorphism in the bidegree $(4, 4 + n_s^{(j)})$ for $5 \leq j \leq 7$ and all $s > 0$. Indeed, due to Theorem 1.7, for each positive integer s , one gets

$$(2.18) \quad \text{Ext}_{\mathcal{A}}^{4, 4+n_s^{(j)}}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \langle g_s \rangle & \text{if } j = 5, \\ \langle h_{s-1}^2 h_{s+3} h_{s+6}, p'_s \rangle & \text{if } j = 6, \\ \langle h_{s-1}^2 h_{s+5}^2, D_3(s) \rangle & \text{if } j = 7. \end{cases}$$

We see that for each integer $s > 0$, the decomposable elements $h_{s-1}^2 h_{s+3} h_{s+6}$ and $h_{s-1}^2 h_{s+5}^2$ are detected by the fourth transfer, $Tr_4(\mathbb{F}_2)$. However, by Theorem 1.2, $g_s \notin \text{Im}(Tr_4(\mathbb{F}_2))$ (see [BHH05]), and by Theorem 1.3, $p'_s \notin \text{Im}(Tr_4(\mathbb{F}_2))$ (see [Hun05]) and $D_3(s) \notin \text{Im}(Tr_4(\mathbb{F}_2))$ (see [Hun05]). So, the degrees $n_s^{(j)}$ are bad and $Tr_4(\mathbb{F}_2)$ is not an epimorphism in those degrees for $5 \leq j \leq 7$ and any $s > 0$. Thus, to prove that $Tr_4(\mathbb{F}_2)$ is a monomorphism in degrees $n_s^{(j)}$, we must demonstrate that Conjecture 1.5 is valid in these degrees $n_s^{(j)}$. Noting that the result for $j = 5$ has already been established by Bruner, Hà, and Hưng in [BHH05]. More explicitly, in [BHH05], the authors stated that the coinvariant space $\mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_s^{(5)}}(V^{\oplus 4})$ is trivial for any $s > 0$. This, combined with the results from (2.18), suggests that Conjecture 1.5 holds in the degrees $n_s^{(5)}$ for all $s > 0$.

We will now demonstrate the proof of the corollary for the cases where j equals 6 and 7.

Proof. It is straightforward to verify that $\mu(n_s^{(6)}) = 4$ for every $s > 1$ and $\mu(n_s^{(7)}) = 4$ for every $s > 2$. Therefore, as a result of Theorem 1.6(ii), the following isomorphisms are obtained:

$$(2.19) \quad \begin{aligned} \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_s^{(6)}}(V^{\oplus 4}) &\cong \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_1^{(6)}}(V^{\oplus 4}), \quad \text{for any } s \geq 2, \\ \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_s^{(7)}}(V^{\oplus 4}) &\cong \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_2^{(7)}}(V^{\oplus 4}), \quad \text{for any } s \geq 3. \end{aligned}$$

On the other hand, according to the proof of Theorem 7.3 in Hưng [Hun05], we have:

$$(2.20) \quad \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_1^{(6)}}(V^{\oplus 4}) = 1 = \dim \mathbb{F}_2 \otimes_{GL(4)} P_{\mathcal{A}} H_{n_2^{(7)}}(V^{\oplus 4}).$$

Then, by the equalities (2.18), (2.19) and (2.20), it follows that Conjecture 1.5 holds true in the degrees $n_s^{(6)}$ and $n_s^{(7)}$ for every positive integer s . Thus, the desired conclusion of the corollary is established. $\square$

The Main Theorem in combination with Corollaries 2.1 and 2.2 yields

Corollary 2.3. *Conjecture 1.5 holds true for all n and so is, Conjecture 1.1 for $Tr_4(\mathbb{F}_2)$.*

Acknowledgments. The author commenced a comprehensive examination of Singer's conjecture of the fourth rank in late 2017 and underwent multiple rounds of revision and improvement to arrive at the present iteration. This substantial project was successfully completed after a duration of five years.

The final version of the paper was published online in the Proceedings of the Royal Society of Edinburgh, Section A (August 23, 2022), DOI: <https://doi.org/10.1017/prm.2022.57>

All results were presented at the "Mathematical Conference on Algebra - Number theory - Geometry and Topology (DAHITO)" held at Thai Nguyen Pedagogical University in October 2021 in Vietnam, as well as at the "27th International Conference of International Academy of Physical Sciences (CONIAPS XXVII) on Recent Advances in Pure and Applied Algebra (RAPAA)" organized by the Department of Mathematics at National Institute of Technology Jamshedpur, India and the International Academy of Physical Sciences (IAPS) in Prayagraj, India in October 2021. The speaker wants to express his sincere appreciation to both the organizers and attendees of these conferences for their meaningful contributions to the productive exchanges that unfolded.

I would also like to extend my heartfelt and grateful appreciation to the anonymous reviewer for his meticulous corrections and valuable feedback on the original version of this work. Especially, the reviewer proposed Conjecture 1.5, which greatly improved the quality of the paper's exposition. Once more, the author wishes to acknowledge and thank him for this insightful suggestion.

Finally, the author feels compelled to express his profound gratitude to Professor Nguyễn Sum, whose unwavering support and guidance were pivotal to the author's successful completion of the PhD thesis from 2015 until the end of 2017. Besides, I would like to show my greatest appreciation to him for the valuable insights and discussions regarding the hit problem of four variables in the spectacular work [Sum15]. These meaningful conversations served as a crucial foundation and a great source of motivation, leading to the completion of this project.

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