A heuristic asymptotic formula for the prime counting function using distribution theory

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Abstract

The aim of this paper is to introduce distribution theory into the field of multiplicative number theory through heuristic methods. This is an endeavour to obtain new insights in the field of number theory by merging two very different fields. In the paper we create a simple but exact formula for the prime counting function using elementary properties of primes. Then we define two distributions $h(z)$ and $H(z)$ which are inspired by the Dirac delta distribution. Using these two distributions we attempt to create an asymptotic formula for the prime counting function and thus create a bridge between distribution theory and multiplicative number theory.

Keywords : Prime counting function, Distribution theory

1 Introduction

An asymptotic formula for the prime counting function is very important in order to find some underlying pattern in the distribution of primes. By using the prime number theorem we can find a useful asymptotic formula for the prime counting function. The prime number theorem states the following

$$\pi(x) \sim \frac{x}{\log x}$$

This approximation has been made better. Now we can write

$$\pi(x) \sim Li(x)$$

Here $Li(x)$ is the logarithmic integral defined as the following

$$Li(x) = \int_2^{\infty} \frac{dx}{\log x}$$

There is an error term in here. If we assume the Riemann hypothesis to be true then we can write the error term as

$$\mathcal{O}(x^{\frac{1}{2}} \log x)$$

A more detailed derivation of this asymptotic formula for the prime counting function can be found in [1]. The derivation of these asymptotic formulas use analytic number theory.

Now we try to derive a different asymptotic formula for the prime counting function without using analytic number theory. For this we will start with a simple but exact heuristic equation for the prime counting function. From that we will derive an asymptotic approximation for the prime counting function by using two step functions. But here we make an attempt to describe the step function using distributions. Then by using properties of these newly defined distributions, we derive a new asymptotic formula for the prime counting function.

2 A heuristic formula for $\pi(x)$ and asymptotic approximation for the odd prime counting function

We are going to approach this by making an observation. We can see by observing the prime numbers that every prime has no other divisor's other than 1 and the number itself and every prime is an odd number except for 2. From this simple fact we can clearly see that the prime counting function can be written as

$$\pi(x) = 1 + \lim_{j \rightarrow 0} \sum_{n=3}^x \prod_{k=2}^{n-1} j^{\lfloor 1 - \{\frac{n}{k}\} \rfloor} \quad (1)$$

where $\{x\}$ is the fractional part of x when $x \in \mathbb{R}$. The other function is the floor function $\lfloor \cdot \rfloor$. This formula for the prime counting function holds for

all $x \in \mathbb{N}$ and $x \geq 3$. This works because $\lfloor 1 - \{\frac{n}{k}\} \rfloor$ becomes 0 for n being prime and becomes 1 for n not being prime. Thus the exponent ensures that the term in the sum becomes zero if n is not prime and becomes 1 if n is prime. Now let's modify the expression using the Dirac delta function. Recall that the Dirac delta function has the following properties

$$\delta(x - x_0) := \begin{cases} 0; & \text{for } x \neq x_0 \\ \infty; & \text{for } x = x_0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) dx = 1$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0)$$

In [2] one can find a better explanation of the properties of the Dirac distribution. Now let's modify equation (1). For this we first notice that the expression inside the sum is always finite and is bounded from above and below and thus by using a special case of the dominated convergence theorem we can bring the limit inside the sum. Then we can write the following

$$\begin{aligned} \pi(x) &= 1 + \sum_{n=3}^x \lim_{j \rightarrow 0} e^{(\log j \sum_{k=2}^{n-1} \lfloor 1 - \{\frac{n}{k}\} \rfloor)} \\ \Rightarrow \pi(x) &= 1 + \sum_{n=3}^x \lim_{j \rightarrow 0} \exp \left(\log j \sum_{k=2}^{n-1} \int_{-\infty}^{\infty} \delta(x - \{\frac{n}{k}\}) H(x) dx \right) \end{aligned}$$

Where we define a new function $H(x)$ in the following manner

$$H(x) := \begin{cases} 1; & \text{for } x=0 \\ 0; & \text{for } x \neq 0 \end{cases}$$

Here $x \in \mathbb{R}$. Now this function is discrete and to give it a continuous form we are going to treat it as a distribution. We have defined the distribution such that

$$\int_{-\infty}^{\infty} H(x) dx = 0$$

And for any continuous function $f(x)$ having compact support

$$\int_{-\infty}^{\infty} f(x) H(x - x_0) dx = \lim_{\gamma \rightarrow \infty} \frac{f(x_0)}{\gamma}$$

Here the R.H.S. only has a non-zero value if and only if $f(x_0)$ is so big that it cancels the effect of γ which tends to infinity. This distribution has a similar measure to the Dirac measure though the measure is defined very differently. Now if we let $\log j = A$ and change the limit in the above expression then we get the following

$$\pi(x) = 1 + \sum_{n=3}^x \lim_{A \rightarrow \infty} \exp \left(-A \sum_{k=2}^{n-1} \int_{-\infty}^{\infty} \delta(x - \{\frac{n}{k}\}) H(x) dx \right)$$

Now let's modify the $H(x)$ distribution and think of it as a probability density function similar to the probability density function of the normal distribution. So let's define

$$H(x) := \lim_{\epsilon \rightarrow \infty} e^{-\epsilon^2 x^2 \pi}$$

Now let's modify the above expression for the prime counting function.

$$\begin{aligned} \pi(x) &= 1 + \sum_{n=3}^x \lim_{A \rightarrow \infty} \exp \left(-A \sum_{k=2}^{n-1} \int_{-\infty}^{\infty} \delta(x - \{\frac{n}{k}\}) \lim_{\epsilon \rightarrow \infty} e^{-\epsilon^2 x^2 \pi} dx \right) \\ \Rightarrow \pi(x) &= 1 + \sum_{n=3}^x \lim_{A \rightarrow \infty} \exp \left(-A \sum_{k=2}^{n-1} \lim_{\epsilon \rightarrow \infty} \int_{-\infty}^{\infty} \delta(x - \{\frac{n}{k}\}) e^{-\epsilon^2 x^2 \pi} dx \right) \\ \Rightarrow \pi(x) &= 1 + \sum_{n=3}^x \lim_{A \rightarrow \infty} \exp \left(-A \sum_{k=2}^{n-1} \lim_{\epsilon \rightarrow \infty} e^{-\epsilon^2 \pi \{\frac{n}{k}\}^2} \right) \end{aligned}$$

Now let's write $n = mk + r_{n,k}$ where $r_{n,k} = n \pmod k$ and m is the largest multiple of k for which this relation holds. Also A , which is $\log j$ and goes to infinity as $j \rightarrow 0$, can be changed with ϵ because both A and ϵ tend to infinity and ϵ is arbitrary so the definition of A doesn't matter. Then the above equation becomes

$$\begin{aligned} \pi(x) &= 1 + \sum_{n=3}^x \lim_{\epsilon \rightarrow \infty} e^{-\epsilon \sum_{k=2}^{n-1} e^{-\epsilon^2 \pi \{\frac{r_{n,k}}{k}\}^2}} \\ \therefore \pi(x) &= 1 + \sum_{n=3}^x \lim_{\epsilon \rightarrow \infty} e^{-\epsilon \sum_{k=2}^{n-1} e^{-\epsilon^2 \pi (\frac{r_{n,k}}{k})^2}} \end{aligned} \tag{2}$$

Now we do some asymptotic approximation. We notice that

$$0 \leq \frac{r_{n,k}}{k} < 1$$

and we can clearly see that

$$-\left(\frac{r_{n,k}}{k}\right)^2 \geq -\left(\frac{\sqrt{n}}{k}\right)^2$$

So in the above equation

$$-\epsilon \sum_{k=2}^{n-1} e^{-\epsilon^2 \pi \left(\frac{r_{n,k}}{k}\right)^2} \leq -\epsilon \sum_{k=2}^{n-1} e^{-\epsilon^2 \pi \left(\frac{\sqrt{n}}{k}\right)^2}$$

So

$$\pi(x) \leq 1 + \sum_{n=3}^x \lim_{\epsilon \rightarrow \infty} \exp \left(-\epsilon \sum_{k=2}^{n-1} e^{-\epsilon^2 \pi \left(\frac{n}{k^2}\right)} \right)$$

Now to get a better result we are going to replace ϵ with $\frac{1}{\sqrt{\pi}} \log t$. So as $\epsilon \rightarrow \infty$ then $t \rightarrow \infty$. After making this substitution we have

$$\pi(x) \leq 1 + \sum_{n=3}^x \lim_{t \rightarrow \infty} \prod_{k=2}^{n-1} t^{-\frac{n \log t}{\sqrt{\pi} k^2}}$$

Let's define a new function $\chi(x) := \pi(x) - 1$. This function only counts the odd primes. Then the above expression becomes

$$\chi(x) \leq \sum_{n=3}^x \lim_{t \rightarrow \infty} \prod_{k=2}^{n-1} t^{-\frac{n \log t}{\sqrt{\pi} k^2}}$$

Now as $x \rightarrow \infty$ we can write the above expression as the following

$$\chi(x) \sim \sum_{n=3}^x \lim_{t \rightarrow \infty} \prod_{k=2}^{n-1} t^{-\frac{n \log t}{\sqrt{\pi} k^2}} \quad (3)$$

3 Definition of a new step function and derivation of the asymptotic formula for the prime counting function

Let's now define a new step function $h(x)$ as the following

$$h(x) := \begin{cases} 0, & \text{for } x \notin \mathbb{Z} \\ 1, & \text{for } x \in \mathbb{Z} \end{cases}$$

We want to specify that this step function is a distribution just like $H(x)$ from the earlier section. Now we notice that the R.H.S. of eq. (3) can be

written in the following manner

$$\sum_{n=3}^x \lim_{t \rightarrow \infty} \prod_{k=2}^{z-1} t^{-\frac{z \log t}{\sqrt{\pi k^2}}} \sim \int_3^x \lim_{t \rightarrow \infty} \prod_{k=2}^{z-1} t^{-\frac{z \log t}{\sqrt{\pi k^2}}} dz \quad (4)$$

Which holds true as $x \rightarrow \infty$. We can now do some modifications of the integral above

$$\int_3^x \lim_{t \rightarrow \infty} \prod_{k=2}^{z-1} t^{-\frac{z \log t}{\sqrt{\pi k^2}}} h(z) dz = \int_3^x \exp \left(- \lim_{t \rightarrow \infty} \sum_{k=2}^{z-1} \frac{z \log^2 t}{\sqrt{\pi k^2}} \right) h(z) dz \quad (5)$$

Now we use the same step function to change the summation inside the exponential of the above integral. Then we get the following

$$\begin{aligned} \sum_{k=2}^{z-1} \frac{z \log^2 t}{\sqrt{\pi k^2}} &= \int_2^{z-1} \frac{z \log^2 t}{\sqrt{\pi p^2}} h(p) dp \\ &= \frac{z \log^2 t}{\sqrt{\pi}} \int_2^{z-1} \frac{h(p)}{p^2} dp \end{aligned}$$

Since by our definition the distribution only has two values and the other function in the integrand is $\frac{1}{p^2}$ which we know is a monotonically decreasing function so we know that the above integral must converge. Now we can write the function inside the exponential in equation (5) as the following

$$\lim_{t \rightarrow \infty} \sum_{k=2}^{z-1} \frac{z \log^2 t}{\sqrt{\pi k^2}} \sim \lim_{t \rightarrow \infty} \frac{z \log^2 t}{\sqrt{\pi}} \sum_{i=2}^{z-2} \int_i^{i+1} \frac{h(p)}{p^2} dp \quad (6)$$

We can now use the fact that $h(p)$ is a distribution and so we can think of it as a really thin bell curve with the peak being in the point where p is an integer and the rest of the area of this bell curve being concentrated around p . The distribution spreads δ amount around the peak. Now if we consider another function being multiplied with this distribution and consider the area under the intersection of the distribution and the function that we have multiplied with the distribution, then we find that the area will be approximately a rectangle. So if we do the integration then we can write it as the sum of the areas of these rectangles. So we can write the R.H.S. of the above expression as

$$\lim_{t \rightarrow \infty} \frac{z \log^2 t}{\sqrt{\pi}} \sum_{i=2}^{z-2} \int_i^{i+1} \frac{h(p)}{p^2} dp = \lim_{\substack{t \rightarrow \infty \\ \delta \rightarrow 0}} \frac{z \log^2 t}{\sqrt{\pi}} \sum_{i=2}^{z-2} \left[\frac{\delta}{(i+1)^2} + \frac{\delta}{i^2} \right]$$

$$\begin{aligned}
&= \lim_{\substack{t \rightarrow \infty \\ \delta \rightarrow 0}} \frac{z\delta \log^2 t}{\sqrt{\pi}} \left[2\zeta(2) - \left(\frac{9}{4} + 2 \sum_{i=z-2}^{\infty} \frac{1}{i^2} \right) \right] \\
&= \lim_{\substack{t \rightarrow \infty \\ \delta \rightarrow 0}} \frac{z\delta \log^2 t}{\sqrt{\pi}} \left[\frac{\pi^2}{3} - \left(\frac{9}{4} + 2 \sum_{i=z-2}^{\infty} \frac{1}{i^2} \right) \right]
\end{aligned}$$

Here we are going to use Abel summation formula to calculate the sum

$$\sum_{i=z-2}^{\infty} \frac{1}{i^2}$$

We know that the Abel summation formula is the following

$$\sum_{x < n \leq y} a(n)\phi(n) = A(y)\phi(y) - A(x)\phi(x) - \int_x^y A(t)\phi'(t)dt$$

Where $A(x) = \sum_{0 \leq n \leq x} a(n)$ and $\phi(x)$ is a continuously differentiable function on $[x, y]$, where $x, y \in \mathbb{N}$ and $x < y$. For our purposes we are only considering Abel's summation formula for $a(n) \in \mathbb{R}$ and $\phi(x) \in \mathbb{R}$, however, we can also expand this formula for $a(n) \in \mathbb{C}$ and $\phi(x) \in \mathbb{C}$. For our particular case $a(n) = 1$ for all $n \in [x, y]$ and thus $A(t) = t$. For our case Abel summation formula can be written as

$$\begin{aligned}
\sum_{z-1 < i}^{\infty} \frac{1}{i^2} &= \frac{1}{1-z} + 2 \int_{z-1}^{\infty} \frac{1}{t^2} dt \\
&= \frac{1}{z-1}
\end{aligned}$$

Here we can clearly see that

$$\sum_{z-1 < i}^{\infty} \frac{1}{i^2} = \sum_{i=z-2}^{\infty} \frac{1}{i^2}$$

So the R.H.S. of equation (6) can be written as the following

$$\lim_{t \rightarrow \infty} \frac{z \log^2 t}{\sqrt{\pi}} \sum_{i=2}^{z-2} \int_i^{i+1} \frac{h(p)}{p^2} dp = \lim_{\substack{t \rightarrow \infty \\ \delta \rightarrow 0}} \frac{z\delta \log^2 t}{\sqrt{\pi}} \left[\frac{\pi^2}{3} - \left(\frac{9}{4} + \frac{2}{z-1} \right) \right]$$

Then we can express equation (5) as the following

$$\int_3^x \lim_{t \rightarrow \infty} \prod_{k=2}^{z-1} t^{-\frac{z \log t}{\sqrt{\pi} k^2}} h(z) dz = \int_3^x \exp \left(- \lim_{\substack{t \rightarrow \infty \\ \delta \rightarrow 0}} \frac{z\delta \log^2 t}{\sqrt{\pi}} \left[\frac{\pi^2}{3} - \left(\frac{9}{4} + \frac{2}{z-1} \right) \right] \right) dz \quad (7)$$

Now, equation (7) can be written as the following

$$\int_3^x \lim_{t \rightarrow \infty} \prod_{k=2}^{z-1} t^{-\frac{z \log t}{\sqrt{\pi} k^2}} h(z) dz = \int_3^x \exp \left(- \lim_{\substack{t \rightarrow \infty \\ \delta \rightarrow 0}} \frac{z \delta \log^2 t}{\sqrt{\pi}} \left[\frac{\pi^2}{3} - \left(\frac{9}{4} + \frac{2}{z-1} \right) \right] \right) dz$$

And from equation (4) as the following

$$\sum_{n=3}^x \lim_{t \rightarrow \infty} \prod_{k=2}^{z-1} t^{-\frac{z \log t}{\sqrt{\pi} k^2}} \sim \int_3^x \exp \left(- \lim_{\substack{t \rightarrow \infty \\ \delta \rightarrow 0}} \frac{z \delta \log^2 t}{\sqrt{\pi}} \left[\frac{\pi^2}{3} - \left(\frac{9}{4} + \frac{2}{z-1} \right) \right] \right) dz \quad (8)$$

Now we know from our definition of $\chi(x)$ that

$$\pi(x) = 1 + \chi(x)$$

So by combining equation (3), (4), and (8) we have

$$\boxed{\pi(x) \sim 1 + \int_3^x \exp \left(- \lim_{\substack{t \rightarrow \infty \\ \delta \rightarrow 0}} \frac{z \delta \log^2 t}{\sqrt{\pi}} \left[\frac{\pi^2}{3} - \left(\frac{9}{4} + \frac{2}{z-1} \right) \right] \right) dz}$$

The boxed equation above is our new asymptotic formula for the prime counting function. Though the actual value of $\lim_{\substack{t \rightarrow \infty \\ \delta \rightarrow 0}} \delta \log^2 t$ is not known but it is sure to approach zero. However, for practical applications the above formula can give a decent result if $\lim_{\substack{t \rightarrow \infty \\ \delta \rightarrow 0}} \delta \log^2 t$ is taken to be a number which has the same number of 0's after the radix point as there are decimal places in x after x 's most significant digit. Though the above formula heuristically should work for when $x \rightarrow \infty$ and then we can use $\lim_{\substack{t \rightarrow \infty \\ \delta \rightarrow 0}} \delta \log^2 t \rightarrow 0$, but for practical applications when $x < \infty$ then the author believes heuristically $\lim_{\substack{t \rightarrow \infty \\ \delta \rightarrow 0}} \delta \log^2 t \rightarrow 0$ should be equal to $\frac{C}{x}$ where C is some real number. Even though the formula above was derived through heuristic methods but the numerical values that it outputs is almost similar and of the same order of magnitude to that found by the formula $\frac{x}{\log x}$ (which has been rigorously proved by the prime number theorem) if an appropriately small value of $\lim_{\substack{t \rightarrow \infty \\ \delta \rightarrow 0}} \delta \log^2 t$ is taken.

4 Remarks

The manner in which we have derived our new asymptotic formula for the prime counting function is purely heuristic and is in need of rigorous proofs.

However, from this method we have shown how concepts from distribution theory can be applied to number theory. We can extend the method to incorporate various other functions that deals with integers. There might also be an avenue to extend these distributions into the field of quantum mechanics and thus creating a connection between multiplicative number theory and quantum mechanics. However, the author is not certain about the last assertion.

Data availability statement: There is no associated data to this manuscript.

Appendix

A Proof that $H(z)$ is a distribution induced by a measure similar to the Dirac measure

Intuitively we can see that $H(z)$ is induced by a measure similar to the Dirac distribution though they are not the same. But we now need to prove it. For this we are going to prove the following theorem.

Theorem A.1. *Let $H(x)$ be defined in the following manner.*

$$H(x - x') := \begin{cases} 1; & \text{for } x=x' \\ 0; & \text{for } x \neq x' \end{cases}$$

Where $x \in \mathbb{R}$ And it has the following properties

$$\int_{-\infty}^{\infty} H(x) dx = 0$$

and for any continuous function $f(x)$ having compact support

$$\int_{-\infty}^{\infty} f(x) H(x - x') dx = \lim_{\gamma \rightarrow \infty} \frac{f(x')}{\gamma}$$

Then $H(x)$ is a distribution induced by the measure $\mu_H(x)$ on the measurable space (X, Σ, μ_H) and $\mu_H : G \rightarrow \bar{\mathbb{R}}$, where $G \subset \Sigma$ and $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. $\mu_H(x)$ is defined in the following way

$$\mu_H(x) = \begin{cases} \frac{1}{\epsilon'}; & \text{for } x=x_0 \\ 0; & \text{for } x \neq x_0 \end{cases}$$

here $x_0 \in G$ and $\epsilon' > \sup_{x \in G} (x)$. Also $\mu_H(x)$ has the following properties

$$\begin{aligned}
1. \quad & \mu_H(\emptyset) = 0 \\
2. \quad & \mu_H\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu_H(A_n)
\end{aligned}$$

Where $\{A_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint measurable sets and $A_n \in \Sigma$ for $n \in \mathbb{N}$. And for $\{A_n\}_{n \in \mathbb{N}}$ being a sequence of measurable sets we have

$$\begin{aligned}
3. \quad & \mu_H\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu_H(A_n) \\
4. \quad & \mu_H(A) + \mu_H(A^c) = \bar{\mathbb{R}}
\end{aligned}$$

and

$$5. \quad \int_X f d\mu_H = \frac{f(x_0)}{\epsilon'}$$

Proof. First we prove that $\mu_H(x)$ is actually measurable. For this to be true the first four properties of $\mu_H(x)$ must actually hold from the definition of $\mu_h(x)$. From the definition of $\mu_H(x)$, the first property follows trivially. Now for the second property can be easily proved in the following way-

If $A_n \notin G$ then $\mu_H(A_n) = 0$ and if $A_n \in G$ then $\mu_H(A_n) = 0$ for $x_0 \notin A_n$ and $\mu_H(A_n) = \frac{1}{\epsilon'}$ for $x_0 \in A_n$. So

$$\mu_H\left(\bigcup_{n=1}^{\infty} A_n\right) = \begin{cases} \frac{1}{\epsilon'}; & \text{for } x_0 \in A_n \\ 0; & \text{for } x_0 \notin A_n \end{cases}$$

Also now if we want to do the sum then we can see that every term in the sum is 0 except when $x_0 \in A_n$, then the sum is equal to $\frac{1}{\epsilon'}$. Meaning

$$\sum_{n=1}^{\infty} \mu_H(A_n) = \begin{cases} \frac{1}{\epsilon'}; & \text{for } x_0 \in A_n \\ 0; & \text{for } x_0 \notin A_n \end{cases}$$

So property (2) holds true and by a similar argument for the case where the sequence $\{A_n\}_{n \in \mathbb{N}}$ is connected then only if $A_i \cup A_j = x_0$ then $\mu_H(\bigcup_{n=1}^{\infty} A_n) = \frac{1}{\epsilon'}$ otherwise $\mu_H(\bigcup_{n=1}^{\infty} A_n) = 0$. But the sum on the other hand is almost always going to be

$$\sum_{n=1}^{\infty} \mu_H(A_n) = \frac{1}{\epsilon'}$$

because for all $A_n \subset \Sigma$, the sequence $\{A_n\}_{n \in \mathbb{N}}$ will almost certainly contain the set element x_0 . And from a similar argument we can see that property

(4) of the measure $\mu_H(x)$ is also true from the definition of $\mu_H(x)$. So $\mu_H(x)$ is actually measurable. Now to prove the fifth property from the definition we need the concept of Lebesgue measure. Actually since this measure by definition is singular, so the Lebesgue measure can not be directly used here but using a similar notion to the Lebesgue measure we can see that from our particular definition of $\mu_H(x)$ we arrive at property (5). Now let's take $X = \mathbb{R}$ and take a Lebesgue measurable subset $C \subset \mathbb{R}$. Then for a smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$ and for the Lebesgue measure $\mu(x)$ we can write the following integral with $H(x)$

$$\begin{aligned} \int_C H(x - x_0) f(x) d\mu(x) &= \int_C f d\mu_H \\ &= f(x_0) \mu_H(C) \end{aligned}$$

Where the last line is a consequence of property (5). Now if we take $f(x) = 1$ then the above expression gives

$$\begin{aligned} \int_C H(x - x_0) d\mu(x) &= \int_C d\mu_H \\ &= \mu_H(C) \end{aligned}$$

From this we can state that $H(x)$ acts similar to the Radon-Nikodym derivative of $\mu_H(x)$ with respect to $\mu(x)$. Now since the measure $\mu_H(x)$ is a point measure so we can say that $H(x)$ is an irregular distribution induced by the measure $\mu_H(x)$. ■

From this proof we can state as a corollary that $h(z)$ is an irregular distribution induced by the measure $\mu_h(z)$ defined in the measure space (X, Σ, μ_h) , where $\mu_h : E \rightarrow \bar{\mathbb{R}}$ and

$$\mu_h(z) = \begin{cases} \frac{1}{\epsilon'}; & \text{for } z \in E \\ 0; & \text{for } z \notin E \end{cases}$$

for $E \subset \Sigma$ and $\epsilon' > \sup_{x \in E} (x)$. In order for $\mu_h(z)$ to induce $h(z)$ we set $E = \mathbb{Z}$.

B Proof that the Dirac delta function acts on $H(z)$ the same way it acts on a continuous function with compact support

In the main paper we have used the Dirac delta function as though it acts on $H(z)$ the same way it acts on a continuous function with compact support. However this is simply not true. So we are going to try to justify our process in the main paper via the following theorem.

Theorem B.1. *Let $\delta(t)$ denote the Dirac distribution. This is defined as*

$$\delta(x - x_0) := \begin{cases} 0; & \text{for } x \neq x_0 \\ \infty; & \text{for } x = x_0 \end{cases}$$

And it has the following properties

1. $\int_{-\infty}^{\infty} \delta(x - x_0) dx = 1$
2. $\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0)$

This Delta distribution is defined as the derivative of the Heaviside step function $\Theta(x)$ for $x \in \mathbb{R}$, where $\Theta(x)$ is defined in the following manner

$$\Theta(x) = \begin{cases} 1; & \text{for } x \geq 0 \\ 0; & \text{for } x < 0 \end{cases}$$

And $H(x)$ is defined as stated earlier. Then

$$\int_{-\infty}^{\infty} \delta(x - x_0) H(x) dx = H(x_0)$$

Proof. For the measure space (X, Σ, μ_H) we choose $X = \mathbb{R}$. Now

$$\begin{aligned} \int_{\mathbb{R}} \delta(x) H(x) dx &= \int_{\mathbb{R}} \left\{ \frac{d}{dx} \Theta(x) \right\} H(x) dx \\ &= - \int_{\mathbb{R}} \Theta(x) \left\{ \frac{d}{dx} H(x) \right\} dx \\ &= - \int_0^{\infty} \frac{d}{dx} H(x) dx \end{aligned}$$

Similarly we can write

$$\int_{\mathbb{R}} \delta(x - x_0) H(x) dx = - \int_{x_0}^{\infty} \frac{d}{dx} H(x) dx$$

$$= H(x_0)$$

The last line comes from the definition of $H(x)$. So we have

$$\int_{-\infty}^{\infty} \delta(x - x_0)H(x)dx = H(x_0)$$

Now this is only true if the distributional derivative of $H(x)$ actually exists. For this let's take a test function $\phi(x)$ that is continuous and has compact support. Then

$$\begin{aligned} \int_{\mathbb{R}} \left\{ \frac{d}{dx} H(x) \right\} \phi(x) dx &= H(\infty)\phi(\infty) - H(-\infty)\phi(-\infty) - \int_{\mathbb{R}} H(x) \left\{ \frac{d}{dx} \phi(x) \right\} dx \\ &= - \int_{\mathbb{R}} H(x) \left\{ \frac{d}{dx} \phi(x) \right\} dx \end{aligned}$$

And this is the definition of the distributional derivative. So the distributional derivative of $H(x)$ actually exists. ■

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