A FORMULA FOR THE r -COLOURED PARTITION FUNCTION IN TERMS OF THE SUM OF DIVISORS FUNCTION AND ITS INVERSE

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ABSTRACT. Let $p_{-r}(n)$ denote the r -coloured partition function, and $\sigma(n) = \sum_{d|n} d$ denote the sum of positive divisors of n . The aim of this note is to prove the following

$$p_{-r}(n) = \theta(n) + \sum_{k=1}^{n-1} \frac{r^{k+1}}{(k+1)!} \sum_{\alpha_1=k}^{n-1} \sum_{\alpha_2=k-1}^{\alpha_1-1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1} \theta(n-\alpha_1)\theta(\alpha_1-\alpha_2) \cdots \theta(\alpha_{k-1}-\alpha_k)\theta(\alpha_k)$$

where $\theta(n) = n^{-1} \sigma(n)$, and its inverse

$$\sigma(n) = n \sum_{r=1}^n \frac{(-1)^{r-1}}{r} \binom{n}{r} p_{-r}(n).$$

1. MAIN RESULTS

Definition 1. [1] A partition of a positive integer n is a finite weakly decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ such that $\sum_{i=1}^r \lambda_i = n$. Let $p(n)$ denote the number of partitions of n .

A partition is said to be r -coloured if each part can occur as r colours. Let $p_{-r}(n)$ denote the number of r coloured partitions of n . The generating function for $p_{-r}(n)$ is given by

$$\sum_{n=0}^{\infty} p_{-r}(n) q^n = E(q)^{-r}$$

where

$$E(q) := \prod_{j=1}^{\infty} (1 - q^j)$$

where $|q| < 1$.

In the following, let $\sigma(n) = \sum_{d|n} d$ denote the sum of divisors of a positive integer n . We first prove the following.

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Theorem 1. For all positive integers $n \geq 2$ we have

$$p_{-r}(n) = \theta(n) + \sum_{k=1}^{n-1} \frac{r^{k+1}}{(k+1)!} \sum_{\alpha_1=k}^{n-1} \sum_{\alpha_2=k-1}^{\alpha_1-1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1} \theta(n-\alpha_1) \theta(\alpha_1-\alpha_2) \cdots \theta(\alpha_{k-1}-\alpha_k) \theta(\alpha_k) \quad (1)$$

where $\theta(n) = n^{-1} \sigma(n)$.

Lemma 1. We have

$$-\log(E(q)) = \sum_{n=1}^{\infty} \theta(n) q^n.$$

Proof. It is easy to see that

$$\begin{aligned} \log(E(q)) &= \sum_{j=1}^{\infty} \log(1 - q^j) \\ &= - \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \frac{q^{lj}}{l} \\ &= - \sum_{n=1}^{\infty} q^n \left(\sum_{d|n} \frac{1}{d} \right). \end{aligned}$$

This completes the proof. $\square$

Definition 2. For n and k non-negative integers, the partial (n, k) th partial Bell polynomials in the variables $x_1, x_2, \dots, x_{n-k+1}$ denoted by $B_{n,k} \equiv B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ [3, p. 206] can be defined by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \mathbb{N} \\ \sum_{i=1}^n i \ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!} \right)^{\ell_i}.$$

Cvijović [2] gives the following formula for calculating these polynomials

$$\begin{aligned} B_{n,k+1} &= \frac{1}{(k+1)!} \underbrace{\sum_{\alpha_1=k}^{n-1} \sum_{\alpha_2=k-1}^{\alpha_1-1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_k \overbrace{\binom{n}{\alpha_1} \binom{\alpha_1}{\alpha_2} \cdots \binom{\alpha_{k-1}}{\alpha_k}}^k \\ &\quad \cdot x_{n-\alpha_1} x_{\alpha_1-\alpha_2} \cdots x_{\alpha_{k-1}-\alpha_k} x_{\alpha_k} \quad (n \geq k+1, k = 1, 2, \dots) \end{aligned} \quad (2)$$

Lemma 2. We have

$$p_{-r}(n) = \frac{1}{n!} \sum_{k=1}^n r^k B_{n,k}(1! \theta(1), 2! \theta(2), \dots, (n-k+1)! \theta(n-k+1)), \quad (3)$$

where $\theta(n) = n^{-1}\sigma(n)$.

Proof. Let $f(q) = e^{rq}$, and $g(q) = -\log(E(q))$. Using Faà di Bruno's formula [3, p. 134] we have

$$\frac{d^n}{dq^n} f(g(q)) = \sum_{k=1}^n f^{(k)}(g(q)) \cdot B_{n,k}(g'(q), g''(q), \dots, g^{(n-k+1)}(q)). \quad (4)$$

Since $f^{(k)}(q) = r^k e^{rq}$ and $g(0) = 1$, letting $q \rightarrow 0$ in the above equation gives us equation (3). $\square$

Combining equations (3) and (2) we can conclude (1).

Now we prove the following.

Theorem 2. *We have*

$$\sigma(n) = n \sum_{r=1}^n \frac{(-1)^{r-1}}{r} \binom{n}{r} p_{-r}(n). \quad (5)$$

Lemma 3. *We have*

$$\theta(n) = \frac{1}{n!} \sum_{k=1}^n (-1)^{k-1} (k-1)! B_{n,k}(1!p(1), 2!p(2), \dots, (n-k+1)!p(n-k+1)) \quad (6)$$

where $p(n) = p_{-1}(n)$ is the partition function.

Proof. Let $f(q) = \log q$, and $g(q) = 1/E(q)$. Then using Faà di Bruno's formula (4) we have

$$\frac{d^n}{dq^n} f(g(q)) = \sum_{k=1}^n f^{(k)}(g(q)) \cdot B_{n,k}(g'(q), g''(q), \dots, g^{(n-k+1)}(q)).$$

Since $f^{(k)}(q) = \frac{(-1)^{k-1}(k-1)!}{q^k}$ and $g(0) = 1$, letting $q \rightarrow 0$ in the above equation gives us equation (6). $\square$

Lemma 4. *We have for positive integers n, k*

$$B_{n,k}(1!p(1), 2!p(2), \dots, (n-k+1)!p(n-k+1)) = \frac{n!}{k!} \sum_{r=1}^k (-1)^{k-r} \binom{k}{r} p_{-r}(n) \quad (7)$$

Proof. We start with the generating function for the partial Bell polynomials as follows

$$\begin{aligned} \sum_{n=k}^{\infty} B_{n,k}(1!p(1), 2!p(2), \dots, (n-k+1)!p(n-k+1)) \frac{q^n}{n!} &= \frac{1}{k!} \left(\sum_{j=1}^{\infty} p(j) q^j \right)^k \\ &= \frac{1}{k!} (E(q)^{-1} - 1)^k \\ &= \frac{1}{k!} \sum_{r=0}^k (-1)^{k-r} \binom{k}{r} E(q)^{-r} \end{aligned}$$

$$= \frac{1}{k!} \sum_{r=0}^k (-1)^{k-r} \binom{k}{r} \sum_{n=0}^{\infty} p_{-r}(n) q^n$$

to conclude the equation (7). □

Proof of Theorem 2. Combining equations (6) and (7) we have

$$\begin{aligned} \sum_{d|n} \frac{1}{d} &= \sum_{k=1}^n \frac{1}{k} \sum_{r=1}^k (-1)^{r-1} \binom{k}{r} p_{-r}(n) \\ &= \sum_{r=1}^n (-1)^{r-1} p_{-r}(n) \sum_{k=r}^n \frac{1}{k} \binom{k}{r} \\ &= \sum_{r=1}^n \frac{(-1)^{r-1}}{r} \binom{n}{r} p_{-r}(n). \end{aligned}$$

Now we can conclude our main result equation (5) after the fact that $\sum_{d|n} \frac{n}{d} = \sum_{j|n} j$. □

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