Journal of Hydraulic Engineering

Discussion of "Exact Analytical Solutions of the Colebrook-White Equation" by Yozo Mikata and Walter S. Walczak, J. Hydraul. Eng. 04015050; doi. 10.1061/(ASCE)HY.1943-7900.0001074

--Manuscript Draft--

Manuscript Number:	HYENG-9876R1
Full Title:	Discussion of "Exact Analytical Solutions of the Colebrook-White Equation" by Yozo Mikata and Walter S. Walczak, J. Hydraul. Eng. 04015050; doi. 10.1061/(ASCE)HY.1943-7900.0001074
Manuscript Region of Origin:	ITALY
Article Type:	Discussion
Corresponding Author:	Dejan Brkic, Ph.D. in Petroleum Eng. European Commission Ispra (VA), ITALY
Corresponding Author E-Mail:	dejanbrkic0611@gmail.com
Order of Authors:	Dejan Brkic, Ph.D. in Petroleum Eng.
Additional Information:	
Question	**Response**
Do your table titles/figure captions cite other sources? If you used a figure/table from another source, written permission for print and online use must be attached in PDF format. Permission letters must state that permission is granted in both forms of media. If you used data from another source to create your own figure/table, the data is adapted and therefore obtaining permission is not required.	No such things.
Estimates for color figures in the printed journal begin at $924. Cost increases depend on the number and size of figures. Do you intend for any figure to be printed in color? If YES, how many and which ones? Please provide a total count and also list them by figure number.	No such color figures.

Cover Letter

Dear Editor,

Please find attached revised version of the Discussion of “Exact Analytical Solutions of the Colebrook-White Equation” by Yozo Mikata and Walter S. Walczak, J. Hydraul. Eng. 04015050; doi. 10.1061/(ASCE)HY.1943-7900.0001074.

Dejan Brkić

Ispra (VA), Italy

April, 14th 2016

Discussion of "Exact Analytical Solutions of the Colebrook-White Equation" by Yozo Mikata and Walter S. Walczak, J. Hydraul. Eng. 04015050; doi. 10.1061/(ASCE)HY.1943-7900.0001074

Dejan Brkić, PhD, Research Fellow; European Commission; DG Joint Research Center (JRC); Institute for Energy and Transport (IET); Energy Security, Systems and Market Unit; Via Enrico Fermi 2749; Ispra (VA); Italy; dejanbrkic0611@gmail.com, dejan.brkic@jrc.ec.europa.eu

This discussion communicates comparisons, practical applications and accuracy check related to the closed-form solutions of the Colebrook-White equation (Colebrook 1939, Colebrook and White 1937) showed in the discussed paper and provides comments and references to already available similar works.

The first exact closed-form solution of the Colebrook-White equation in terms of Lambert W-function appears in Keady (1998) [Eq. (9) of the discussed paper]. Sonnad and Goudar (2004), confirmed also by Brkić (2012a), show constraints for the use of Keady's equation because it contains exponential form as the argument of the Lambert W-function (as defined in Hayes (2005), "W" represents the Lambert function). Very high numerical values caused by exponential form in Keady's case very often make impossible calculation using computers (these values are too high to be processed by registers of memory and processor unit). These cases are associated in engineering practice with high speed of flow through very rough pipes (higher numerical values of the Reynolds (R) and higher values the relative roughness of inner pipe surface (ε/D) occurred simultaneously in pairs). Rollman and Spindler (2015), referred to general properties of the Lambert W-function (Corless et al. 1996), give their solution in order to avoid such exponential form in the transformed Colebrook-White equation (similar as Eq. (14) of the discussed paper). To overwhelm such inconvenience Brkić (2011a) also transforms the Colebrook-White equation using Lambert W-function avoiding exponential term (1):

$$\frac{1}{\sqrt{f}}=-2\cdot\log_{10}\left(\frac{2\cdot 2.51\cdot W\left[\frac{\ln 10}{2}\cdot\frac{R}{2.51}\right]}{R\cdot\ln 10}+\frac{\varepsilon}{3.71\cdot D}\right)=-2\cdot\log_{10}\left(10^{\frac{-W\left[\frac{\ln 10}{2}\cdot\frac{R}{2.51}\right]}{\ln 10}}+\frac{\varepsilon}{3.71\cdot D}\right) \quad (1)$$

To develop (1), Brkić (2011a) use limiting case of the Colebrook-White equation under smooth pipe law of flow as showed in Appendix I of the discussed paper, where $\varepsilon/D\rightarrow 0$ (Goudar and Sonnad 2003, Sonnad and Goudar 2006, Brkić 2011b; 2012b). To make Eq. (1) more applicable for engineering practice, Brkić (2011c) replaces Lambert W-function with approximate calculus. Further for the purpose of this discussion, numerical values of the parameters from this explicit approximation are optimized using genetic algorithms developed by Ćojbašić and Brkić (2013) where the relative error, $\delta(\%)=(|f-f_0|/f_0)\cdot 100\%$, decreases to 1.28% (before optimization it was 2.2% and 3.16%, respectively referred to Eq. (2)):

$$\left.\begin{aligned}&\frac{1}{\sqrt{f}}\approx-2.013\cdot\log_{10}\left(\frac{2.261\cdot A}{R}+\frac{1}{3.71}\cdot\frac{\varepsilon}{D}\right)\approx-2.013\cdot\log_{10}\left(10^{-0.43\cdot A}+\frac{1}{3.71}\cdot\frac{\varepsilon}{D}\right)\\&A\approx\ln\frac{R}{2.479\cdot\ln\left(\frac{1.1\cdot R}{\ln\left(1+1.1\cdot R\right)}\right)}\end{aligned}\right\} \quad (2)$$

Regarding soft computation techniques such as optimization through genetic algorithms, it should be noted that the Colebrook-White equation can be simulated very accurately using Artificial Neural Networks as showed in Brkić and Ćojbašić (2016).

The *n*th formula is developed (Eq. (26) of the discussed paper) using similar approach as in Brkić (2011a, 2012a). In this case, the Boyd's "shifted" Lambert W-function (Boyd 1998) is used which is noted as the Y function the discussed paper. In addition to the solution based on Boyd's function, Brkić (2011a, 2012a,c) presents some further solutions based on works of Barry et al. (2000) and

Winitzki (2003). Also, Brkić (2011a, 2012a,c) uses the series expansion of the Lambert W-function in a similar way as it is done for of Y function (Eq. (44) of the discussed paper).

Approximations of the Colebrook-White equation based on the *n*th formula, Eq (21) of the discussed paper, are given by (3).

$$\left.\begin{array}{r}\frac{1}{\sqrt{f_{n=1}}}\approx\frac{2}{\ln 10}\cdot\left[\ln\left(\frac{\ln 10}{2}\cdot\frac{R}{2.51}\right)-\ln\left(x_1-\ln x_1\right)\right]\\ \frac{1}{\sqrt{f_{n=2}}}\approx\frac{2}{\ln 10}\cdot\left[\ln\left(\frac{\ln 10}{2}\cdot\frac{R}{2.51}\right)-\ln\left(x_1-\ln\left(x_1-\ln x_1\right)\right)\right]\\ \frac{1}{\sqrt{f_{n=3}}}\approx\frac{2}{\ln 10}\cdot\left[\ln\left(\frac{\ln 10}{2}\cdot\frac{R}{2.51}\right)-\ln\left(x_1-\ln\left(x_1-\ln\left(x_1-\ln x_1\right)\right)\right)\right]\\ \vdots\\ \frac{1}{\sqrt{f_n}}\approx\frac{2}{\ln 10}\cdot\left[\ln\left(\frac{\ln 10}{2}\cdot\frac{R}{2.51}\right)-\ln\left(x_1-\ln\left(x_1-\ln\left(x_1-\underbrace{\cdots-\ln x_1\cdots}_{n-1}\right)\right)\right)\right]\\ x_1=\ln\left(\frac{\ln 10}{2}\cdot\frac{R}{2.51}\right)+\frac{\ln 10}{2}\cdot\frac{R}{2.51}\cdot\frac{\varepsilon}{3.71\cdot D}\end{array}\right\} \quad (3)$$

Using methodology from Brkić (2011d), the maximal percentage relative error is evaluated. It is 1.1877% for n=1, 0.1826% for n=2, 0.0278% for n=3, 0.004249% for n=4, $6.48\cdot10^{-4}$% for n=4, $9.89\cdot10^{-5}$% for n=5, etc. The error aggregates in the zone of small values of the Reynolds number (R) and relative roughness (ε/D) and it decreases with geometric progression. On the other hand, these approximations are computationally demanded since they contain many logarithmic terms (Clamond 2009, Giustolisi et al. 2011).

Notation

The following symbols are used in this discussion:

R – Reynolds number (dimensionless)

ε/D – Relative roughness of inner pipe surface (dimensionless)

f – Darcy (Moody) flow friction factor (dimensionless)

"=" – exactly equivalent to the Colebrook-White equation

"≈"– approximately equivalent to the Colebrook-White equation

f_0 – Darcy (Moody) flow friction factor (dimensionless); obtained from the Colebrook-White equation using iterative procedure and hence treated as accurate

$\delta(\%)=(|f-f_0|/f_0)\cdot 100\%$ - relative error (%)

W – Lambert function

X_1-parameter defined by Eq. (14) of the discussed paper (here in Eq. 3)

A-auxiliary term used in Eq. 2

'ln' denotes the natural log function

Disclaimer: Neither the European Commission nor any person acting on behalf of it is responsible for the use which might be made of this publication.

References:

Barry, D.A., Parlange, J.Y., Li, L., Prommer, H., Cunningham, C.J. and Stagnitti, F. (2000). "Analytical approximations for real values of the Lambert W-function." Math. Comput. Simulat 53(1/2), 95-103. doi. 10.1016/S0378-4754(00)00172-5

Boyd, J.P. (1998). "Global approximations to the principal real-valued branch of the Lambert W-function." Appl. Math. Lett. 11(6), 27-31. doi. 10.1016/S0893-9659(98)00097-4

Brkić, D. (2011a). "W solutions of the CW equation for flow friction." Appl. Math. Lett. 24(8), 1379-1383. doi. 10.1016/j.aml.2011.03.014

Brkić, D. (2011b). "New explicit correlations for turbulent flow friction factor." Nucl. Eng. Des. 241(9), 4055-4059. doi. 10.1016/j.nucengdes.2011.07.042

Brkić, D. (2011c). "An explicit approximation of the Colebrook equation for fluid flow friction factor." Petrol. Sci. Tech. 29(15), 1596-1602. doi. 10.1080/10916461003620453

Brkić, D. (2011d). “Review of explicit approximations to the Colebrook relation for flow friction.” J. Petrol. Sci. Eng. 77(1), 34-48. doi. 10.1016/j.petrol.2011.02.006

Brkić, D. (2012a). “Comparison of the Lambert W-function based solutions to the Colebrook equation.” Eng. Computation 29(6), 617-630. doi. 10.1108/02644401211246337

Brkić, D. (2012b). “Can pipes be actually really that smooth?” Int. J. Refrig. 35(1), 209-215. doi. 10.1016/j.ijrefrig.2011.09.012

Brkić, D. (2012c). “Determining friction factors in turbulent pipe flow.” Chem. Eng. (New York) 119(3), 34-39. From: www.chemengonline.com/determining-friction-factors-in-turbulent-pipe-flow/?printmode=1

Brkić, D., and Ćojbašić, Ž. (2016). “Intelligent flow friction estimation.” Comput. Intell. Neurosci. Article ID 5242596, 10 pages, doi. 10.1155/2016/5242596

Clamond, D. (2009). “Efficient resolution of the Colebrook equation.” Ind. Eng. Chem. Res., 48(7), 3665–3671. doi. 10.1021/ie801626g

Ćojbašić, Ž, and Brkić, D. (2013). “Very accurate explicit approximations for calculation of the Colebrook friction factor.” Int. J. Mech. Sci. 67, 10-13. doi. 10.1016/j.ijmecsci.2012.11.017

Colebrook, C.F. (1939). “Turbulent flow in pipes with particular reference to the transition region between the smooth and rough pipe laws.” J. Inst. Civil. Eng. (London) 11(4), 133-156. doi. 10.1680/ijoti.1939.13150

Colebrook, C.F., and White, C.M. (1937). “Experiments with fluid friction in roughened pipes.” Proc. Roy. Soc. Ser. A Math. Phys. Sci. 161(906), 367-381. doi. 10.1098/rspa.1937.0150

Corless, R.M., Gonnet, G.H., Hare, D.E.G., Jeffrey, D.J., and Knuth, D.E. (1996), “On the Lambert W-function.” Adv. Comput. Math. 5(1), 329-359. doi. 10.1007/BF02124750

Giustolisi, O., Berardi, L., and Walski, T.M. (2011). “Some explicit formulations of Colebrook-White friction factor considering accuracy vs. computational speed.” J. Hydroinf. 13(3), 401-418. doi. 10.2166/hydro.2010.098

Goudar, C.T. and Sonnad, J.R. (2003), “Explicit friction factor correlation for turbulent flow in smooth pipes”, Ind. Eng. Chem. Res. 42(12), 2878-2880. doi. 10.1021/ie0300676

Hayes, B. (2005) “Why W?” Am. Sci., 93(2), 104-108. doi. 10.1511/2005.2.104

Keady, G. (1998). “Colebrook-White formula for pipe flow.” J. Hydraul. Eng. ASCE 124(1), 96-97. doi. 10.1061/(ASCE)0733-9429(1998)124:1(96)

Rollmann, P., and Spindler, K. (2015). “Explicit representation of the implicit Colebrook–White equation.” Case Studies in Thermal Engineering 5, 41-47. doi. 10.1016/j.csite.2014.12.001

Sonnad, J.R., and Goudar, C.T. (2004). “Constraints for using Lambert W function-based explicit Colebrook–White equation.” J. Hydraul. Eng. ASCE 130(9), 929–931. doi. 10.1061/(ASCE)0733-9429(2004)130:9(929)

Sonnad, J.R., and Goudar, C.T. (2006). “Turbulent flow friction factor calculation using a mathematically exact alternative to the Colebrook–White equation.” J. Hydraul. Eng. ASCE 132(8), 863-867. doi. 10.1061/(ASCE)0733-9429(2006)132:8(863)

Winitzki, S. (2003), “Uniformapproximations for transcendental functions”, in Kumar,V.,Gavrilova, M.L., Jeng, C., Tan, K. and L’Ecuyer, P. (Eds), Computational Science and Its Applications – ICCSA 2003, International Conference, Montreal, Canada, May 18-21, Proceedings, Lecture Notes in Computer Science, Part 1, 2667, Springer, Berlin, pp. 780-9.

ASCE Authorship, Originality, and Copyright Transfer Agreement

Publication Title: Journal of Hydraulic Engineering

Manuscript Title: Discussion of "Exact Analytical Solutions of the Colebrook-White Equation" by Yozo Mikata and Walter S. Walczak, J. Hydraul. Eng. 04015050; doi. 10.1061/(ASCE)HY.1943-7900.0001074

Author(s) – **Names, postal addresses, and e-mail addresses of all authors**

Dejan Brkić, PhD, Research Scientific Officer;
European Commission; DG Joint Research Center (JRC); Institute for Energy and Transport (IET); Energy Security,
Systems and Market Unit; Offshore Oil and Gas Group,
Via Enrico Fermi 2749; Ispra (VA); Italy;
dejanbrkic0611@gmail.com, dejan.brkic@jrc.ec.europa.eu

I. Authorship Responsibility

To protect the integrity of authorship, only people who have significantly contributed to the research or project and manuscript preparation shall be listed as coauthors. The corresponding author attests to the fact that anyone named as a coauthor has seen the final version of the manuscript and has agreed to its submission for publication. Deceased persons who meet the criteria for coauthorship shall be included, with a footnote reporting date of death. No fictitious name shall be given as an author or coauthor. An author who submits a manuscript for publication accepts responsibility for having properly included all, and only, qualified coauthors.

I, the corresponding author, confirm that the authors listed on the manuscript are aware of their authorship status and qualify to be authors on the manuscript according to the guidelines above.

Dejan Brkić	[signature]	OCTOBER 21ST, 2015
Print Name	Signature	Date

II. Originality of Content

ASCE respects the copyright ownership of other publishers. ASCE requires authors to obtain permission from the copyright holder to reproduce any material that (1) they did not create themselves and/or (2) has been previously published, to include the authors' own work for which copyright was transferred to an entity other than ASCE. Each author has a responsibility to identify materials that require permission by including a citation in the figure or table caption or in extracted text. Materials re-used from an open access repository or in the public domain must still include a citation and URL, if applicable. At the time of submission, authors must provide verification that the copyright owner will permit re-use by a commercial publisher in print and electronic forms with worldwide distribution. For Conference Proceeding manuscripts submitted through the ASCE online submission system, authors are asked to verify that they have permission to re-use content where applicable. Written permissions are not required at submission but must be provided to ASCE if requested. Regardless of acceptance, no manuscript or part of a manuscript will be published by ASCE without proper verification of all necessary permissions to re-use. ASCE accepts no responsibility for verifying permissions provided by the author. Any breach of copyright will result in retraction of the published manuscript.

I, the corresponding author, confirm that all of the content, figures (drawings, charts, photographs, etc.), and tables in the submitted work are either original work created by the authors listed on the manuscript or work for which permission to re-use has been obtained from the creator. For any figures, tables, or text blocks exceeding 100 words from a journal article or 500 words from a book, written permission from the copyright holder has been obtained and supplied with the submission.

Dejan Brkić	[signature]	OCTOBER 21ST, 2015
Print name	Signature	Date

III. Copyright Transfer

ASCE requires that authors or their agents assign copyright to ASCE for all original content published by ASCE. The author(s) warrant(s) that the above-cited manuscript is the original work of the author(s) and has never been published in its present form.

The undersigned, with the consent of all authors, hereby transfers, to the extent that there is copyright to be transferred, the exclusive copyright interest in the above-cited manuscript (subsequently called the "work") in this and all subsequent editions of the work (to include closures and errata), and in derivatives, translations, or ancillaries, in English and in foreign translations, in all formats and media of expression now known or later developed, including electronic, to the American Society of Civil Engineers subject to the following:

- The undersigned author and all coauthors retain the right to revise, adapt, prepare derivative works, present orally, or distribute the work, provided that all such use is for the personal noncommercial benefit of the author(s) and is consistent with any prior contractual agreement between the undersigned and/or coauthors and their employer(s).
- No proprietary right other than copyright is claimed by ASCE.
- If the manuscript is not accepted for publication by ASCE or is withdrawn by the author prior to publication (online or in print), or if the author opts for open-access publishing during production (journals only), this transfer will be null and void.
- Authors may post a PDF of the ASCE-published version of their work on their employers' ***Intranet*** with password protection. The following statement must appear with the work: "This material may be downloaded for personal use only. Any other use requires prior permission of the American Society of Civil Engineers."
- Authors may post the ***final draft*** of their work on open, unrestricted Internet sites or deposit it in an institutional repository when the draft contains a link to the published version at www.ascelibrary.org. "Final draft" means the version submitted to ASCE after peer review and prior to copyediting or other ASCE production activities; it does not include the copyedited version, the page proof, a PDF, or full-text HTML of the published version.

Exceptions to the Copyright Transfer policy exist in the following circumstances. Check the appropriate box below to indicate whether you are claiming an exception:

☐ **U.S. GOVERNMENT EMPLOYEES:** Work prepared by U.S. Government employees in their official capacities is not subject to copyright in the United States. Such authors must place their work in the public domain, meaning that it can be freely copied, republished, or redistributed. In order for the work to be placed in the public domain, ALL AUTHORS must be official U.S. Government employees. If at least one author is not a U.S. Government employee, copyright must be transferred to ASCE by that author.

☐ **CROWN GOVERNMENT COPYRIGHT:** Whereby a work is prepared by officers of the Crown Government in their official capacities, the Crown Government reserves its own copyright under national law. If ALL AUTHORS on the manuscript are Crown Government employees, copyright cannot be transferred to ASCE; however, ASCE is given the following nonexclusive rights: (1) to use, print, and/or publish in any language and any format, print and electronic, the above-mentioned work or any part thereof, provided that the name of the author and the Crown Government affiliation is clearly indicated; (2) to grant the same rights to others to print or publish the work; and (3) to collect royalty fees. ALL AUTHORS must be official Crown Government employees in order to claim this exemption in its entirety. If at least one author is not a Crown Government employee, copyright must be transferred to ASCE by that author.

☐ **WORK-FOR-HIRE:** Privately employed authors who have prepared works in their official capacity as employees must also transfer copyright to ASCE; however, their employer retains the rights to revise, adapt, prepare derivative works, publish, reprint, reproduce, and distribute the work provided that such use is for the promotion of its business enterprise and does not imply the endorsement of ASCE. In this instance, an authorized agent from the authors' employer must sign the form below.

☐ **U.S. GOVERNMENT CONTRACTORS:** Work prepared by authors under a contract for the U.S. Government (e.g., U.S. Government labs) may or may not be subject to copyright transfer. Authors must refer to their contractor agreement. For works that qualify as U.S. Government works by a contractor, ASCE acknowledges that the U.S. Government retains a nonexclusive, paid-up, irrevocable, worldwide license to publish or reproduce this work for U.S. Government purposes only. This policy DOES NOT apply to work created with U.S. Government grants.

I, the corresponding author, acting with consent of all authors listed on the manuscript, hereby transfer copyright or claim exemption to transfer copyright of the work as indicated above to the American Society of Civil Engineers.

Dejan Brkić

Print Name of Author or Agent

OCTOBER 21ST 2015

Signature of Author of Agent | Date

More information regarding the policies of ASCE can be found at http://www.asce.org/authorsandeditors

Answers to the reviewer's comment:

The text is thoroughly checked and grammatical errors are hopefully corrected.