Credit Default Swap Model

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ABSTRACT

This article presents a model for valuing a credit default swap (CDS) contract that is affected by the credit risks of the buyer, seller and reference entity. Many people in market believe that, if a CDS is fully collateralized, there is no risk of failure to pay. Collateral posting regimes are originally designed and utilized for interest rate swap (IRS), but there are many reasons to be concerned about the success of collateral posting in offsetting the risk of CDS contracts. First, the value of CDS contracts tends to move very suddenly with big jumps, whereas the price movements of *IRS* contracts are far smoother and less volatile than *CDS* prices. Second, CDS spreads can widen very rapidly. Third, CDS contracts have many more risk factors than *IRS* contracts. In fact, our model shows that full collateralization *cannot* eliminate counterparty risk completely for a CDS contract.

Key Words: valuation model; credit risk modeling; collateralization; correlation, CDS.

1 Introduction

This paper presents a new framework for valuing defaultable financial contracts with or without collateral arrangements. The framework characterizes default dependencies exogenously, and models collateral processes directly based on collateral agreements. For brevity we focus on CDS contracts, but many of the points we make are equally applicable to other derivatives. CDS has trilateral credit risk, where three parties – buyer, seller and reference entity – are defaultable.

In general, a CDS contract is used to transfer the credit risk of a reference entity from one party to another. The risk circularity that transfers one type of risk (reference credit risk) into another (counterparty credit risk) within the CDS market is a concern for financial stability. Some people claim that the CDS market has increased financial contagion or even propose an outright ban on these instruments.

The standard CDS pricing model in the market assumes that there is no counterparty risk. Although this oversimplified model may be accepted in normal market conditions, its reliability in times of distress has recently been questioned. In fact, counterparty risk has become one of the most dangerous threats to the CDS market.

There is a significant increase in the use of collateral for CDS after the recent financial crises. Many people believe that, if a CDS is fully collateralized, there is no risk of failure to pay. There are many reasons to be concerned about the success of collateral posting in offsetting the risk of CDS contracts. First, the value of CDS contracts tends to move very suddenly with big jumps, whereas the price movements of *IRS* contracts are far smoother and less volatile than *CDS* prices. Second, CDS spreads can widen very rapidly. Third, CDS contracts have many more risk factors than *IRS* contracts. In fact, our model shows that full collateralization *cannot* eliminate counterparty risk completely for a CDS contract.

The rest of this paper is organized as follows: Pricing multilateral defaultable financial contract is elaborated on in Section 2; numerical results are provided in Section 3; the conclusions are presented in Section 4. All proofs and some detailed derivations are contained in the appendices.

2 Pricing CDS Subject to Multiple Credit Risks

We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$ satisfying the usual conditions, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ -algebra, $\mathcal{P}$ denotes a probability measure, and $\{\mathcal{F}_t\}_{t \geq 0}$ denotes a filtration.

In the reduced-form approach, the stopping (or default) time τ_i of firm i is modeled as a Cox arrival process (also known as a doubly stochastic Poisson process) whose first jump occurs at default and is defined by,

$$\tau_i = \inf \left\{ t : \int_0^t h_i(s, Z_s) ds \geq H_i \right\} \quad (1)$$

where $h_i(t)$ or $h_i(t, Z_t)$ denotes the stochastic hazard rate or arrival intensity dependent on an exogenous common state Z_t , and H_i is a unit exponential random variable independent of Z_t .

It is well-known that the survival probability from time t to s in this framework is defined by

$$p_i(t, s) := P_i(\tau > s \mid \tau > t, Z_t) = \exp \left(- \int_t^s h_i(u) du \right) \quad (2a)$$

The default probability for the period (t, s) in this framework is given by

$$q_i(t, s) := P_i(\tau \leq s \mid \tau > t, Z_t) = 1 - p_i(t, s) = 1 - \exp \left(- \int_t^s h_i(u) du \right) \quad (2b)$$

The interest in the financial industry for the modeling and pricing of multilateral defaultable instruments arises mainly in two respects: in the management of credit risk at a portfolio level and in the valuation of credit derivatives. Central to the valuation and risk management of credit derivatives and risky portfolios is the problem of default relationship.

Let us discuss a three-party case first. A CDS is a good example of a trilateral defaultable instrument where the three parties are counterparties A , B and reference entity C . In a standard CDS contract one party purchases credit protection from another party, to cover the loss of the face value of a reference entity following a credit event. The protection buyer makes periodic payments to the seller until the maturity date or until a credit event occurs. A credit event usually requires a final accrual payment by the buyer and a loss protection payment by the protection seller. The protection payment is equal to the difference between par and the price of the cheapest to deliver (CTD) asset of the reference entity on the face value of the protection.

A CDS is normally used to transfer the credit risk of a reference entity between two counterparties. The contract reduces the credit risk of the reference entity but gives rise to another form of risk: *counterparty risk*. Since the dealers are highly concentrated within a small group, any of them may be too big to fail. The interconnected nature, with dealers being tied to each other through chains of OTC derivatives, results in increased contagion risk. Due to its concentration and interconnectedness, the CDS market seems to pose a systemic risk to financial market stability. In fact, the CDS is blamed for playing a pivotal role in the collapse of Lehman Brothers and the disintegration of AIG.

For years, a widespread practice in the market has been to mark CDS to market without taking the counterparty risk into account. The realization that even the most prestigious investment banks could go bankrupt has shattered the foundation of the practice. It is wiser to face frankly the real complexities of pricing a CDS than to indulge in simplifications that have proved treacherous. For some time now it has been realized that, in order to value a CDS properly, counterparty effects have to be taken into account.

Let A denote the protection buyer, B denote the protection seller and C denote the reference entity. The binomial default rule considers only two possible states: default or survival. Therefore, the default indicator Y_j for firm j ($j = A$ or B or C) follows a Bernoulli distribution, which takes value 1 with default probability q_j , and value 0 with survival probability p_j . The marginal default distributions can be determined by the reduced-form models. The joint distributions of a multivariate Bernoulli variable can be

easily obtained via the marginal distributions by introducing extra correlations. The joint probability representations of a trivariate Bernoulli distribution (see Teugels (1990)) are given by

$$p_{000} := P(Y_A = 0, Y_B = 0, Y_C = 0) = p_A p_B p_C + p_C \sigma_{AB} + p_B \sigma_{AC} + p_A \sigma_{BC} - \theta_{ABC} \quad (3a)$$

$$p_{100} := P(Y_A = 1, Y_B = 0, Y_C = 0) = q_A p_B p_C - p_C \sigma_{AB} - p_B \sigma_{AC} + q_A \sigma_{BC} + \theta_{ABC} \quad (3b)$$

$$p_{010} := P(Y_A = 0, Y_B = 1, Y_C = 0) = p_A q_B p_C - p_C \sigma_{AB} + q_B \sigma_{AC} - p_A \sigma_{BC} + \theta_{ABC} \quad (3c)$$

$$p_{001} := P(Y_A = 0, Y_B = 0, Y_C = 1) = p_A p_B q_C + q_C \sigma_{AB} - p_B \sigma_{AC} - p_A \sigma_{BC} + \theta_{ABC} \quad (3d)$$

$$p_{110} := P(Y_A = 1, Y_B = 1, Y_C = 0) = q_A q_B p_C + p_C \sigma_{AB} - q_B \sigma_{AC} - q_A \sigma_{BC} - \theta_{ABC} \quad (3e)$$

$$p_{101} := P(Y_A = 1, Y_B = 0, Y_C = 1) = q_A p_B q_C - q_C \sigma_{AB} + p_B \sigma_{AC} - q_A \sigma_{BC} - \theta_{ABC} \quad (3f)$$

$$p_{011} := P(Y_A = 0, Y_B = 1, Y_C = 1) = p_A q_B q_C - q_C \sigma_{AB} - q_B \sigma_{AC} + p_A \sigma_{BC} - \theta_{ABC} \quad (3g)$$

$$p_{111} := P(Y_A = 1, Y_B = 1, Y_C = 1) = q_A q_B q_C + q_C \sigma_{AB} + q_B \sigma_{AC} + q_A \sigma_{BC} + \theta_{ABC} \quad (3h)$$

where

$$\theta_{ABC} := E((Y_A - q_A)(Y_B - q_B)(Y_C - q_C)) \quad (3i)$$

Equation (3) tells us that the joint probability distribution of three defaultable parties depends not only on the bivariate statistical relationships of all pair-wise combinations (e.g., σ_{ij}) but also on the trivariate statistical relationship (e.g., θ_{ABC}). θ_{ABC} was first defined by Deardorff (1982) as *comvariance*, who use it to correlate three random variables that are the value of commodity net imports/exports, factor intensity, and factor abundance in international trading.

We introduce the concept of *comvariance* into credit risk modeling arena to exploit any statistical relationship among multiple random variables. Furthermore, we define a new statistic, *comrelation*, as a scaled version of comvariance (just like correlation is a scaled version of covariance) as follows:

Definition 1: For three random variables X_A , X_B , and X_C , let μ_A , μ_B , and μ_C denote the means of X_A , X_B , and X_C . The comrelation of X_A , X_B , and X_C is defined by

$$\zeta_{ABC} = \frac{E[(X_A - \mu_A)(X_B - \mu_B)(X_C - \mu_C)]}{\sqrt[3]{E|X_A - \mu_A|^3 \times E|X_B - \mu_B|^3 \times E|X_C - \mu_C|^3}} \quad (4)$$

According to the Holder inequality, we have

$$\begin{aligned} |E[(X_A - \mu_A)(X_B - \mu_B)(X_C - \mu_C)]| &\leq E|(X_A - \mu_A)(X_B - \mu_B)(X_C - \mu_C)| \\ &\leq \sqrt[3]{E|X_A - \mu_A|^3 \times E|X_B - \mu_B|^3 \times E|X_C - \mu_C|^3} \end{aligned} \quad (5)$$

Obviously, the correlation is in the range of $[-1, 1]$. Given the correlation, Equation (3i) can be rewritten as

$$\begin{aligned} \theta_{ABC} &:= E((Y_A - q_A)(Y_B - q_B)(X_C - q_C)) = \zeta_{ABC} \sqrt[3]{E|Y_A - q_A|^3 \times E|Y_B - q_B|^3 \times E|Y_C - q_C|^3} \\ &= \zeta_{ABC} \sqrt[3]{p_A q_A (p_A^2 + q_A^2) p_B q_B (p_B^2 + q_B^2) p_C q_C (p_C^2 + q_C^2)} \end{aligned} \quad (6)$$

where $E(Y_j) = q_j$ and $E|Y_j - q_j|^3 = p_j q_j (p_j^2 + q_j^2)$, $j=A, B, \text{ or } C$.

If we have a series of n measurements of X_A , X_B , and X_C written as x_{Ai} , x_{Bi} and x_{Ci} where $i = 1, 2, \dots, n$, the sample correlation coefficient can be obtained as:

$$\zeta_{ABC} = \frac{\sum_{i=1}^n (x_{Ai} - \mu_A)(x_{Bi} - \mu_B)(x_{Ci} - \mu_C)}{\sqrt[3]{\sum_{i=1}^n |x_{Ai} - \mu_A|^3 \times \sum_{i=1}^n |x_{Bi} - \mu_B|^3 \times \sum_{i=1}^n |x_{Ci} - \mu_C|^3}} \quad (7)$$

More generally, we define the correlation in the context of n random variables as

Definition 2: For n random variables $X_1, X_2, \dots, X_n$, let μ_i denote the mean of X_i where $i=1, \dots, n$.

The correlation of $X_1, X_2, \dots, X_n$ is defined as

$$\zeta_{12\dots n} = \frac{E[(X_1 - \mu_1)(X_2 - \mu_2) \cdots (X_n - \mu_n)]}{\sqrt[n]{E|X_1 - \mu_1|^n \times E|X_2 - \mu_2|^n \cdots E|X_n - \mu_n|^n}} \quad (8)$$

Correlation is just a specific case of correlation where $n = 2$. Again, the correlation $\zeta_{12\dots n}$ is in the range of $[-1, 1]$ according to the Holder inequality.

2.1 Risky valuation without collateralization

Recovery assumptions are important for pricing credit derivatives. If the reference entity under a CDS contract defaults, the best assumption, as pointed out by J. P. Morgan (1999), is that the recovered

value equals the recovery rate times the face value plus accrued interest¹. In other words, the recovery of par value assumption is a better fit upon the default of the reference entity, whereas the recovery of market value assumption is a more suitable choice in the event of a counterparty default².

We have the following proposition.

Proposition 1: *The value of the CDS is given by*

$$V(t) = \sum_{i=1}^m E \left[\left(\prod_{j=0}^{i-1} O(T_j, T_{j+1}) \right) X_i | \mathcal{F}_t \right] + \sum_{i=1}^m E \left[\left(\prod_{j=0}^{i-2} O(T_j, T_{j+1}) \right) \Omega(T_{i-1}, T_i) R(T_{i-1}, T_i) | \mathcal{F}_t \right] \quad (9a)$$

where $t = T_0$ and

$$O(T_j, T_{j+1}) = 1_{(V(T_{j+1}) + X_{j+1}) \geq 0} \phi_B(T_j, T_{j+1}) + 1_{(V(T_{j+1}) + X_{j+1}) < 0} \phi_A(T_j, T_{j+1}) \quad (9b)$$

$$\begin{aligned} \phi_A(T_j, T_{j+1}) = & \{ p_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) + q_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \phi_A(T_{j+1}) \\ & + p_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \bar{\phi}_A(T_{j+1}) + q_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \phi_{AB}(T_{j+1}) \\ & + p_C(T_j, T_{j+1}) \sigma_{AB}(T_j, T_{j+1}) (1 - \phi_A(T_{j+1}) - \bar{\phi}_A(T_{j+1}) + \phi_{AB}(T_{j+1})) \\ & + \sigma_{AC}(T_j, T_{j+1}) [p_B(T_j, T_{j+1}) (1 - \phi_A(T_{j+1})) + q_B(T_j, T_{j+1}) (\bar{\phi}_A(T_{j+1}) - \phi_{AB}(T_{j+1}))] \\ & + \sigma_{BC}(T_j, T_{j+1}) [p_A(T_j, T_{j+1}) (1 - \bar{\phi}_A(T_{j+1})) + q_A(T_j, T_{j+1}) (\phi_A(T_{j+1}) - \phi_{AB}(T_{j+1}))] \\ & + \theta_{ABC}(T_j, T_{j+1}) (-1 + \bar{\phi}_A(T_{j+1}) - \phi_{AB}(T_{j+1}) + \phi_A(T_{j+1})) \} D(T_j, T_{j+1}) \end{aligned} \quad (9c)$$

$$\begin{aligned} \phi_B(T_j, T_{j+1}) = & \{ p_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) + q_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \bar{\phi}_B(T_{j+1}) \\ & + p_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \phi_B(T_{j+1}) + q_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \phi_{AB}(T_{j+1}) \\ & + p_C(T_j, T_{j+1}) \sigma_{AB}(T_j, T_{j+1}) (1 - \phi_B(T_{j+1}) - \bar{\phi}_B(T_{j+1}) + \phi_{AB}(T_{j+1})) \\ & + \sigma_{AC}(T_j, T_{j+1}) [p_B(T_j, T_{j+1}) (1 - \bar{\phi}_B(T_{j+1})) + q_B(T_j, T_{j+1}) (\phi_B(T_{j+1}) - \phi_{AB}(T_{j+1}))] \\ & + \sigma_{BC}(T_j, T_{j+1}) [p_A(T_j, T_{j+1}) (1 - \phi_B(T_j)) + q_A(T_j, T_{j+1}) (\bar{\phi}_B(T_{j+1}) - \phi_{AB}(T_{j+1}))] \\ & + \theta(T_j, T_{j+1}) (-1 + \bar{\phi}_B(T_{j+1}) - \phi_{AB}(T_{j+1}) + \phi_B(T_{j+1})) \} D(T_j, T_{j+1}) \end{aligned} \quad (9d)$$

¹ In the market, there is an average accrual premium assumption, i.e., the average accrued premium is half the full premium due to be paid at the end of the premium.

² Three different recovery models exist in the literature. The default payoff is either i) a fraction of par (Madan and Unal (1998)), ii) a fraction of an equivalent default-free bond (Jarrow and Turnbull (1995)), or iii) a fraction of market value (Duffie and Singleton (1999)).

$$\begin{aligned}
\Omega(T_j, T_{j+1}) = & \{ p_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) q_C(T_j, T_{j+1}) + q_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) q_C(T_j, T_{j+1}) \bar{\varphi}_B(T_{j+1}) \\
& + p_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) q_C(T_j, T_{j+1}) \varphi_B(T_{j+1}) + q_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) q_C(T_j, T_{j+1}) \varphi_{AB}(T_{j+1}) \\
& + q_C(T_j, T_{j+1}) \sigma_{AB}(T_j, T_{j+1}) (1 - \varphi_B(T_{j+1}) - \bar{\varphi}_B(T_{j+1}) + \varphi_{AB}(T_{j+1})) \\
& - \sigma_{AC}(T_j, T_{j+1}) [p_B(T_j, T_{j+1}) (1 - \bar{\varphi}_B(T_{j+1})) + q_B(T_j, T_{j+1}) (\varphi_B(T_{j+1}) - \varphi_{AB}(T_{j+1}))] \\
& - \sigma_{BC}(T_j, T_{j+1}) [p_A(T_j, T_{j+1}) (1 - \varphi_B(T_{j+1})) + q_A(T_j, T_{j+1}) (\bar{\varphi}_B(T_{j+1}) - \varphi_{AB}(T_{j+1}))] \\
& + \theta_{ABC}(T_j, T_{j+1}) (1 - \bar{\varphi}_B(T_{j+1}) + \varphi_{AB}(T_{j+1}) - \varphi_B(T_{j+1})) \} D(T_j, T_{j+1})
\end{aligned} \tag{9e}$$

where $R(T_j, T_{j+1}) = (N(1 - \varphi_C(T_{j+1})) - \alpha(T_j, T_{j+1}))$, $\alpha(T_j, T_{j+1}) = sN\delta(T_s, T)/2$, and $X_i = -sN\delta(T_j, T_{j+1})$.

Proof: See the Appendix.

We may think of $O(t, T)$ as the risk-adjusted discount factor for the premium and $\Omega(t, T)$ as the risk-adjusted discount factor for the default payment. Proposition 1 says that the pricing process of a multiple-payment instrument has a backward nature since there is no way of knowing which risk-adjusted discounting rate should be used without knowledge of the future value. Only on the maturity date, the value of an instrument and the decision strategy are clear. Therefore, the evaluation must be done in a backward fashion, working from the final payment date towards the present.

Proposition 1 provides a general form for pricing a CDS. Applying it to a particular situation in which we assume that counterparties *A* and *B* are default-free, i.e., $p_j = 1$, $q_j = 0$, $\rho_{kl} = 0$, and $\varsigma_{ABC} = 0$, where $j=A$ or B and $k, l=A, B$, or C , we derive the following corollary.

Corollary 1: *If counterparties A and B are default-free, the value of the CDS is given by*

$$\begin{aligned}
V(t) = & \sum_{i=1}^m E \left[\left(\prod_{j=0}^{i-1} O(T_j, T_{j+1}) \right) X_i \middle| \mathcal{F}_t \right] + \sum_{i=1}^m E \left[\left(\prod_{j=0}^{i-2} O(T_j, T_{j+1}) \right) \Omega(T_{i-1}, T_i) R(T_{i-1}, T_i) \middle| \mathcal{F}_t \right] \\
= & \sum_{i=1}^m E [D(t, T_i) p_C(t, T_i) X_i \middle| \mathcal{F}_t] + \sum_{i=1}^m E [D(t, T_i) p_C(t, T_{i-1}) q_C(T_{i-1}, T_i) R(T_{i-1}, T_i) \middle| \mathcal{F}_t]
\end{aligned} \tag{10}$$

where $O(T_{i-1}, T_i) = D(T_{i-1}, T_i) p_C(T_{i-1}, T_i)$; $\Omega(T_{i-1}, T_i) = D(T_{i-1}, T_i) q_C(T_{i-1}, T_i)$.

The proof of this corollary becomes straightforward according to Proposition 1 by setting $\rho_{kl} = 0$,

$\varphi_{AB} = 0$, $\varsigma_{ABC} = 0$, $p_j = 1$, $q_j = 0$, $p_C(t, T_i) = \prod_{g=0}^{i-1} p(T_g, T_{g+1})$, and $D(t, T_i) = \prod_{g=0}^{i-1} D(T_g, T_{g+1})$.

If we further assume that the discount factor and the default probability of the reference entity are uncorrelated and the recovery rate φ_C is constant, we have

Corollary 2: Assume that i) counterparties A and B are default-free, ii) the discount factor and the default probability of the reference entity are uncorrelated; iii) the recovery rate φ_c is constant; the value of the CDS is given by

$$V(t) = \sum_{i=1}^m P(t, T_i) \bar{p}_C(t, T_{i-1}) \bar{q}_C(T_{i-1}, T_i) (N(1 - \varphi_c) - \alpha(T_{i-1}, T_i)) - \sum_{i=1}^m P(t, T_i) \bar{p}_c(t, T_i) sN \delta(T_{i-1}, T_i) \quad (11)$$

where $P(t, T_i) = E[D(t, T_i) | \mathcal{F}_t]$ denotes the bond price, $\bar{p}_c(t, T_i) = E[p_c(t, T_i) | \mathcal{F}_t]$, $\bar{q}_c(t, T_i) = 1 - \bar{p}_c(t, T_i)$, $\bar{p}(t, T_{i-1}) \bar{q}(T_{i-1}, T_i) = \bar{p}(t, T_{i-1}) - \bar{p}(t, T_i)$.

This corollary is easily proved according to Corollary 1 by setting $E[XY | \mathcal{F}_t] = E[X | \mathcal{F}_t] E[Y | \mathcal{F}_t]$ when X and Y are uncorrelated. Corollary 2 is the formula for pricing CDS in the market.

Our methodology can be extended to the cases where the number of parties $n \geq 4$. A generating function for the (probability) joint distribution (see details in Teugels (1990)) of n -variate Bernoulli can be expressed as

$$p^{(n)} = \begin{bmatrix} p_n & -1 \\ q_n & 1 \end{bmatrix} \otimes \begin{bmatrix} p_{n-1} & -1 \\ q_{n-1} & 1 \end{bmatrix} \otimes \dots \otimes \begin{bmatrix} p_1 & -1 \\ q_1 & 1 \end{bmatrix} \sigma^{(n)} \quad (12)$$

where $\otimes$ denotes the Kronecker product; $p^{(n)} = \{p_k^{(n)}\}$ and $\sigma^{(n)} = \{\sigma_k^{(n)}\}$ are vectors containing 2^n components: $p_k^{(n)} = p_{k_1, k_2, \dots, k_n}$, $k = 1 + \sum_{i=1}^n k_i 2^{i-1}$, $k_i \in \{0, 1\}$; $\sigma_k^{(n)} = \sigma_{k_1, k_2, \dots, k_n} = E\left(\prod_{i=1}^n (Y_i - q_i)^{k_i}\right)$.

2.2 Risky valuation with collateralization

Collateralization is the most important and widely used technique in practice to mitigate credit risk. The posting of collateral is regulated by the Credit Support Annex (CSA) that specifies a variety of terms including the threshold, the independent amount, and the minimum transfer amount (MTA), etc. The threshold is the unsecured credit exposure that a party is willing to bear. The minimum transfer amount is the smallest amount of collateral that can be transferred. The independent amount plays the same role as the initial margin (or haircuts).

The collateral amount posted at time t is given by

$$C(t) = \begin{cases} V(t) - H(t) & \text{if } V(t) > H(t) \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

where $H(t)$ is the collateral threshold. In particular, $H(t) = 0$ corresponds to full-collateralization³; $H > 0$ represents partial/under-collateralization; and $H < 0$ is associated with over-collateralization. Full collateralization becomes increasingly popular at the transaction level. In this paper, we focus on full collateralization only, i.e., $C(t) = V(t)$.

According to the ISDA (2013), almost all CDSs are fully collateralized. Many people believe that full collateralization can eliminate counterparty risk completely for CDS.

Collateral posting regimes are originally designed and utilized for bilateral risk products, e.g., IRS, but there are many reasons to be concerned about the success of collateral posting in offsetting the risks of CDS contracts. First, the values of CDS contracts tend to move very suddenly with big jumps, whereas the price movements of IRS contracts are far smoother and less volatile than CDS prices. Second, CDS spreads can widen very rapidly. The amount of collateral that one party is required to provide at short notice may, in some cases, be close to the notional amount of the CDS and may therefore exceed that party's short-term liquidity capacity, thereby triggering a liquidity crisis. Third, CDS contracts have many more risk factors than IRS contracts.

We assume that a CDS is fully collateralized, i.e., the posting of collateral is equal to the amount of the current *MTM* value: $C(t) = V(t)$. For a discrete one-period (t, u) economy, there are several possible states at time u : i) A , B , and C survive with probability p_{000} . The instrument value is equal to the market value $V(u)$; ii) A and B survive, but C defaults with probability p_{001} . The instrument value is the default payment $R(u)$; iii) For the remaining cases, either or both counterparties A and B default. The instrument

³ There are three types of collateralization: Full-collateralization is a process where the posting of collateral is equal to the current *MTM* value. Partial/under-collateralization is a process where the posting of collateral is less than the current *MTM* value. Over-collateralization is a process where the posting of collateral is greater than the current *MTM* value.

value is the future value of the collateral $V(t)/D(t, u)$ (Here we consider the time value of money only).

The value of the collateralized instrument at time t is the discounted expectation of all the payoffs and is given by

$$\begin{aligned} V(t) &= E\{D(t, u)[p_{000}(t, u)V(u) + p_{001}(t, u)R(u) \\ &\quad + (p_{100}(t, u) + p_{010}(t, u) + p_{110}(t, u) + p_{101}(t, u) + p_{011}(t, u) + p_{111}(t, u))V(t)/D(t, u)]|\mathcal{F}_t\} \quad (14a) \\ &= E\{[D(t, u)(p_{000}(t, u)V(s) + p_{001}(t, u)R(u)) + (1 - p_{000}(t, u) - p_{001}(t, u))V(t)]|\mathcal{F}_t\} \end{aligned}$$

or

$$\begin{aligned} &E[(p_A(t, u)p_B(t, u) + \sigma_{AB}(t, u))|\mathcal{F}_t]V(t) \\ &= E\{[D(t, u)(p_C(t, u)V(u) + q_C(t, u)R(u))(p_A(t, u)p_B(t, u) + \sigma_{AB}(t, u)) \\ &\quad + D(t, u)(V(u) - R(u))(p_B(t, u)\sigma_{AC}(t, u) + p_A(t, u)\sigma_{BC}(t, u) - \theta_{ABC}(t, u))]\mathcal{F}_t\} \quad (14b) \end{aligned}$$

If we assume that $(p_A(t, u)p_B(t, u) + \sigma_{AB}(t, u))$ and $D(t, u)(p_C(t, u)V(u) + q_C(t, u)R(u))$ are uncorrelated, we have

$$V(t) = V^F(t) + \xi_{ABC}(t, u)/\psi_{AB}(t, u) \quad (15a)$$

where

$$V^F(t) = E\{D(t, u)[p_C(t, u)V(u) + q_C(t, u)R(u)]|\mathcal{F}_t\} \quad (15b)$$

$$\psi_{AB}(t, u) = E\{[p_A(t, u)p_B(t, u) + \sigma_{AB}(t, u)]|\mathcal{F}_t\} \quad (15c)$$

$$\xi_{ABC}(t, u) = E\{[D(t, u)(p_B(t, u)\sigma_{AC}(t, u) + p_A(t, u)\sigma_{BC}(t, u) - \theta_{ABC}(t, u))(V(u) - R(u))]\mathcal{F}_t\} \quad (15d)$$

The first term $V^F(t)$ in equation (15) is the counterparty-risk-free value of the CDS and the second term is the exposure left over under full collateralization, which can be substantial.

Proposition 2: *If a CDS is fully collateralized, the risky value of the CDS is NOT equal to the counterparty-risk-free value, as shown in equation (15).*

Proposition 2 or equation (15) provides a theoretical explanation for the failure of full collateralization in the CDS market. It tells us that under full collateralization the risky value is in general not equal to the counterparty-risk-free value except in one of the following situations: i) the market value

is equal to the default payment, i.e., $V(u) = R(u)$; ii) firms A , B , and C have independent credit risks, i.e., $\rho_{ij}=0$ and $\zeta_{ABC}=0$; or iii) the following relationship holds $p_B\sigma_{AC} + p_A\sigma_{BC} = \theta_{ABC}$.

3 Numerical Results

The current (spot) market data are shown in Table 1 provide by FinPricing (2018). Assume that the reference entity C has an “A+200bps” credit quality throughout this subsection. The 5-year counterparty-risk-free CDS premium is 0.027 (equals the 5-year ‘A’ rated CDS spread in Table 1 plus 200 basis points).

Since the payoffs of a CDS are mainly determined by credit events, we need to characterize the evolution of the hazard rates. Here we choose the *Cox-Ingersoll-Ross* (CIR) model. The CIR process has been widely used in the literature of credit risk and is given by

$$dh_t = a(b - h_t)dt + \sigma\sqrt{h_t}dW_t \quad (16)$$

where a denotes the mean reversion speed, b denotes the long-term mean, and σ denotes the volatility.

Table 1: Current/spot market data

This table displays the current (spot) market data used for all calculations in this paper, including the term structure of continuously compounded interest rates, the term structure of A-rated breakeven CDS spreads, and the curve of at-the-money caplet volatilities.

Term (days)	31	91	182	365	548	730	1095	1825	2555	3650	5475
Interest Rate	0.002	0.002	0.002	0.004	0.007	0.010	0.016	0.024	0.030	0.035	0.040
	8	7	9	3	1	2		9	6	5	5
Credit Spread	0.004	0.004	0.004	0.004	0.004	0.005	0.005	0.007	0.007	0.009	0.010
	2	2	2	5	9	2	8		9	1	6
Caplet Volatility	0.326	0.331	0.337	0.350	0.364	0.377	0.308	0.247	0.214	0.167	0.163
	7		6	9	1	3		3	1	8	4

Table 2: Risk-neutral parameters for CIR model

This table presents the risk-neutral parameters that are calibrated to the current market shown in Table 1. ‘A+100bps’ represents a ‘100 basis points’ parallel shift in the A-rated CDS spreads.

Credit Quality	A	A+100bps	A+200bps	A+300bps
Long-Term Mean α	0.035	0.056	0.077	0.099
Mean Reverting Speed b	0.14	0.18	0.25	0.36
Volatility σ	0.022	0.028	0.039	0.056

The calibrated parameters are shown in table 2. We assume that interest rates are deterministic and select the regression-based Monte-Carlo simulation (see Longstaff and Schwartz (2001)) to perform risky valuation.

We first assume that counterparties A , B , and reference entity C have independent default risks, i.e., $\rho_{AB} = \rho_{AC} = \rho_{BC} = \varphi_{AB} = \zeta_{ABC} = 0$, and examine the following cases: i) B is risk-free and A is risky; and ii) A is risk-free and B is risky. We simulate the hazard rates using the CIR model and then determine the appropriate discount factors according to Proposition 1. Finally we calculate the prices via the regression-based Monte-Carlo method. The results are shown in Table 3 and 4.

Table 3: Impact of the credit quality of the protection buyer on CDS premia

This table shows how the CDS premium increases as the credit quality of party A decreases. The 1st data column represents the counterparty-risk-free results. For the remaining columns, we assume that party B is risk-free and party A is risky. ‘A+100bps’ represents a ‘100 basis points’ parallel shift in the A-rated CDS spreads. The results in the row ‘Difference from Risk-Free’ = current CDS premium – counterparty-risk-free CDS premium.

Credit Quality	Party A	-	A	A+100bps	A+200bps	A+300bps
	Party B	-	-	-	-	-
CDS premium		0.027	0.02703	0.02708	0.02713	0.02717
Difference from Risk-Free		0	0.003%	0.008%	0.013%	0.017%

Table 4: Impact of the credit quality of the protection seller on CDS premia

This table shows the decrease in the CDS premium with the credit quality of party *B*. The 1st data column represents the counterparty-risk-free results. For the remaining columns, we assume that party *A* is risk-free and party *B* is risky. ‘A+100bps’ represents a ‘100 basis points’ parallel shift in the A-rated CDS spreads. The results in the row ‘Difference from Risk-Free’ = current CDS premium – counterparty-risk-free CDS premium.

Credit Quality	Party A Party B	-	-	-	-	-
		-	A	A+100bps	A+200bps	A+300bps
CDS premium		0.027	0.02695	0.02687	0.0268	0.02672
Difference from Risk-Free		0.00%	-0.005%	-0.013%	-0.020%	-0.028%

From table 3 and 4, we find that a credit spread of about 100 basis points maps into a CDS premium of about 0.4 basis points for counterparty *A* and about -0.7 basis points for counterparty *B*. The credit impact on the CDS premia is approximately linear. As would be expected, i) the dealer’s credit quality has a larger impact on CDS premia than the investor’s credit quality; ii) the higher the investor’s credit risk, the higher the premium that the dealer charges; iii) the higher the dealer’s credit risk, the lower the premium that the dealer asks. Without considering default correlations and comrelations, we find that, in general, the impact of counterparty risk on CDS premia is relatively small.

Figure 1: Impact of default correlations and comrelation on CDS premia

Next, we study the sensitivity of CDS premia to changes in the joint credit quality of associated parties. Sensitivity analysis is a very popular way in finance to find out how the value and risk of an instrument/portfolio changes if risk factors change.

Assume $\rho_{AB}=0.5$. The impact diagrams of the default correlations and comrelation are shown in Figure 1. From this graph, we can draw the following conclusions: First, the CDS premium and the default correlations/comrelation have a negative relation. Intuitively, a protection seller who is positively correlated with the reference entity (a wrong way risk) should charge a lower premium for selling credit protection. Next, the impacts of the default correlations and comrelation are approximately linear. Finally, the sensitivity slopes of the CDS premium to the default correlations and comrelation are -0.06 to ρ_{AB} ; -0.09 to ρ_{AC} ; -53 to ρ_{BC} ; and -14 to ζ_{ABC} . Slope measures the rate of change in the premium as a result of a change in the default dependence. For instance, a slope of -53 implies that the CDS premium would have to decrease by 53 basis points when a default correlation/comrelation changes from 0 to 1.

As the absolute value of the slope increases, so does the sensitivity. The results illustrate that ρ_{BC} has the largest effect on CDS premia. The second biggest one is ζ_{ABC} . The impacts of ρ_{AB} and ρ_{AC} are very small. In particular, the effect of the comrelation is substantial and has never been studied before.

4 Conclusion

This article presents a new valuation framework for pricing financial instruments subject to credit risk. In particular, we focus on modeling default relationships.

Our research shows that accounting for default correlations and comrelations becomes important, especially under market stress. The existing valuation models in the credit derivatives market, which take into account only pair-wise default correlations, may underestimate credit risk and may be inappropriate.

We study the sensitivity of the price of a defaultable instrument to changes in the joint credit quality of the parties. For instance, our analysis shows that the effect of default dependence on CDS premia from large to small is the correlation between the protection seller and the reference entity, the comrelation, the correlation between the protection buyer and the reference entity, and the correlation between the protection buyer and the protection seller.

We conclude that collateralization designed to mitigate counterparty risk works well for financial instruments subject to bilateral credit risk, but fails for ones subject to multilateral credit risk.

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