Credit Risk and Collateral Study

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ABSTRACT

This article presents a solution for pricing derivatives subject to credit risk and collateralization. We study the quantitative relationship between credit default swap (CDS) premia and the credit quality of counterparties and reference entities. The credit impact on the CDS premia is approximately linear. We find that the dealer's credit quality has a larger impact on CDS premia than the investor's credit quality. We also find that the higher the investor's credit risk, the higher the premium that the dealer charges. Our study shows that the higher the dealer's credit risk, the lower the premium that the dealer asks. In general, the impact of counterparty risk on CDS premia is relatively small.

Key Words: derivative pricing; credit risk modeling; collateralization; correlation, CDS.

1 Introduction

There are two primary types of models that attempt to describe default processes in the literature: structural models and reduced-form (or intensity) models. Many practitioners in the credit trading arena have tended to gravitate toward the reduced-form models given their mathematical tractability.

The main drawback of the conditionally independent assumption of the reduced-form models is that the range of default correlations that can be achieved is typically too low when compared with empirical default correlations (see Das et al. (2007)). The responses to correct this weakness can be generally classified into two categories: endogenous default relationship approaches and exogenous default relationship approaches.

The exogenous approaches (see Li (2000), Laurent and Gregory (2005), Hull and White (2004), Brigo et al. (2011), etc) attempt to link marginal default probability distributions to the joint default probability distribution through some external functions. Due to their simplicity in use, practitioners lean toward the exogenous ones.

Collateralization as one of the primary credit risk mitigation techniques becomes increasingly important and widespread in derivatives transactions. According to the ISDA (2013), 73.7% of all OTC derivatives trades (cleared and non-cleared) are subject to collateral agreements. For large firms, the figure is 80.7%. On an asset class basis, 83.0% of all CDS transactions and 79.2% of all fixed income transactions are collateralized. For large firms, the figures are 96.3% and 89.4%, respectively. Previous studies on collateralization include Johannes and Sundareshan (2007), Fujii and Takahashi (2012), Piterbarg (2010), Bielecki, et al (2013) and Hull and White (2014), Xiao (2017), etc.

In general, a CDS contract is used to transfer the credit risk of a reference entity from one party to another. The risk circularity that transfers one type of risk (reference credit risk) into another (counterparty credit risk) within the CDS market is a concern for financial stability. Some people claim that the CDS market has increased financial contagion or even propose an outright ban on these instruments.

The standard CDS pricing model in the market assumes that there is no counterparty risk. Although this oversimplified model may be accepted in normal market conditions, its reliability in times of distress has recently been questioned. In fact, counterparty risk has become one of the most dangerous threats to the CDS market. For some time now it has been realized that, in order to value a CDS properly, counterparty effects have to be taken into account (see ECB (2009)).

There is a significant increase in the use of collateral for CDS after the recent financial crises. Many people believe that, if a CDS is fully collateralized, there is no risk of failure to pay. Collateral posting regimes are originally designed and utilized for bilateral risk products, e.g., interest rate swap (IRS), but there are many reasons to be concerned about the success of collateral posting in offsetting the risk of CDS contracts. First, the value of CDS contracts tends to move very suddenly with big jumps, whereas the price movements of *IRS* contracts are far smoother and less volatile than *CDS* prices. Second, CDS spreads can widen very rapidly. Third, CDS contracts have many more risk factors than *IRS* contracts. In fact, our model shows that full collateralization *cannot* eliminate counterparty risk completely for a CDS contract.

2 Pricing Financial Contracts Subject to Multilateral Credit Risk

We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$ satisfying the usual conditions, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ -algebra, $\mathcal{P}$ denotes a probability measure, and $\{\mathcal{F}_t\}_{t \geq 0}$ denotes a filtration.

It is well-known that the survival probability from time t to s in this framework is defined by

$$p_i(t, s) := P_i(\tau > s \mid \tau > t, Z_t) = \exp\left(-\int_t^s h_i(u) du\right) \quad (2a)$$

The default probability for the period (t, s) in this framework is given by

$$q_i(t, s) := P_i(\tau \leq s \mid \tau > t, Z_t) = 1 - p_i(t, s) = 1 - \exp\left(-\int_t^s h_i(u) du\right) \quad (2b)$$

There is ample evidence that corporate defaults are correlated. The default of a firm's counterparty might affect its own default probability. Thus, default correlation/dependence occurs due to the counterparty relations.

Let us discuss a three-party case first. A CDS is a good example of a trilateral defaultable instrument where the three parties are counterparties A , B and reference entity C . In a standard CDS contract one party purchases credit protection from another party, to cover the loss of the face value of a reference entity following a credit event. The protection buyer makes periodic payments to the seller until the maturity date or until a credit event occurs. A credit event usually requires a final accrual payment by the buyer and a loss protection payment by the protection seller. The protection payment is equal to the difference between par and the price of the cheapest to deliver (CTD) asset of the reference entity on the face value of the protection.

For years, a widespread practice in the market has been to mark CDS to market without taking the counterparty risk into account. The realization that even the most prestigious investment banks could go bankrupt has shattered the foundation of the practice. It is wiser to face frankly the real complexities of pricing a CDS than to indulge in simplifications that have proved treacherous. For some time now it has been realized that, in order to value a CDS properly, counterparty effects have to be taken into account.

The joint probability representations of a trivariate Bernoulli distribution (see Teugels (1990)) are given by

$$p_{000} := P(Y_A = 0, Y_B = 0, Y_C = 0) = p_A p_B p_C + p_C \sigma_{AB} + p_B \sigma_{AC} + p_A \sigma_{BC} - \theta_{ABC} \quad (3a)$$

$$p_{100} := P(Y_A = 1, Y_B = 0, Y_C = 0) = q_A p_B p_C - p_C \sigma_{AB} - p_B \sigma_{AC} + q_A \sigma_{BC} + \theta_{ABC} \quad (3b)$$

$$p_{010} := P(Y_A = 0, Y_B = 1, Y_C = 0) = p_A q_B p_C - p_C \sigma_{AB} + q_B \sigma_{AC} - p_A \sigma_{BC} + \theta_{ABC} \quad (3c)$$

$$p_{001} := P(Y_A = 0, Y_B = 0, Y_C = 1) = p_A p_B q_C + q_C \sigma_{AB} - p_B \sigma_{AC} - p_A \sigma_{BC} + \theta_{ABC} \quad (3d)$$

$$p_{110} := P(Y_A = 1, Y_B = 1, Y_C = 0) = q_A q_B p_C + p_C \sigma_{AB} - q_B \sigma_{AC} - q_A \sigma_{BC} - \theta_{ABC} \quad (3e)$$

$$p_{101} := P(Y_A = 1, Y_B = 0, Y_C = 1) = q_A p_B q_C - q_C \sigma_{AB} + p_B \sigma_{AC} - q_A \sigma_{BC} - \theta_{ABC} \quad (3f)$$

$$p_{011} := P(Y_A = 0, Y_B = 1, Y_C = 1) = p_A q_B q_C - q_C \sigma_{AB} - q_B \sigma_{AC} + p_A \sigma_{BC} - \theta_{ABC} \quad (3g)$$

$$p_{111} := P(Y_A = 1, Y_B = 1, Y_C = 1) = q_A q_B q_C + q_C \sigma_{AB} + q_B \sigma_{AC} + q_A \sigma_{BC} + \theta_{ABC} \quad (3h)$$

where

$$\theta_{ABC} := E((Y_A - q_A)(Y_B - q_B)(Y_C - q_C)) \quad (3i)$$

Equation (3) tells us that the joint probability distribution of three defaultable parties depends not only on the bivariate statistical relationships of all pair-wise combinations (e.g., σ_{ij}) but also on the trivariate statistical relationship (e.g., θ_{ABC}). θ_{ABC} was first defined by Deardorff (1982) as *comvariance*, who use it to correlate three random variables that are the value of commodity net imports/exports, factor intensity, and factor abundance in international trading.

Definition 1: For three random variables X_A , X_B , and X_C , let μ_A , μ_B , and μ_C denote the means of X_A , X_B , and X_C . The comrelation of X_A , X_B , and X_C is defined by

$$\zeta_{ABC} = \frac{E[(X_A - \mu_A)(X_B - \mu_B)(X_C - \mu_C)]}{\sqrt[3]{E|X_A - \mu_A|^3 \times E|X_B - \mu_B|^3 \times E|X_C - \mu_C|^3}} \quad (4)$$

According to the Holder inequality, we have

$$\begin{aligned} |E((X_A - \mu_A)(X_B - \mu_B)(X_C - \mu_C))| &\leq E|(X_A - \mu_A)(X_B - \mu_B)(X_C - \mu_C)| \\ &\leq \sqrt[3]{E|X_A - \mu_A|^3 \times E|X_B - \mu_B|^3 \times E|X_C - \mu_C|^3} \end{aligned} \quad (5)$$

Obviously, the comrelation is in the range of [-1, 1]. Given the comrelation, Equation (3i) can be rewritten as

$$\begin{aligned} \theta_{ABC} &:= E((Y_A - q_A)(Y_B - q_B)(Y_C - q_C)) = \zeta_{ABC} \sqrt[3]{E|Y_A - q_A|^3 \times E|Y_B - q_B|^3 \times E|Y_C - q_C|^3} \\ &= \zeta_{ABC} \sqrt[3]{p_A q_A (p_A^2 + q_A^2) p_B q_B (p_B^2 + q_B^2) p_C q_C (p_C^2 + q_C^2)} \end{aligned} \quad (6)$$

where $E(Y_j) = q_j$ and $E|Y_j - q_j|^3 = p_j q_j (p_j^2 + q_j^2)$, $j=A, B, \text{ or } C$.

If we have a series of n measurements of X_A , X_B , and X_C written as x_{Ai} , x_{Bi} and x_{Ci} where $i = 1, 2, \dots, n$, the sample comrelation coefficient can be obtained as:

$$\zeta_{ABC} = \frac{\sum_{i=1}^n (x_{Ai} - \mu_A)(x_{Bi} - \mu_B)(x_{Ci} - \mu_C)}{\sqrt[3]{\sum_{i=1}^n |x_{Ai} - \mu_A|^3 \times \sum_{i=1}^n |x_{Bi} - \mu_B|^3 \times \sum_{i=1}^n |x_{Ci} - \mu_C|^3}} \quad (7)$$

More generally, we define the *comrelation* in the context of n random variables as

Definition 2: For n random variables $X_1, X_2, \dots, X_n$, let μ_i denote the mean of X_i where $i=1, \dots, n$. The *comrelation* of $X_1, X_2, \dots, X_n$ is defined as

$$\zeta_{12\dots n} = \frac{E[(X_1 - \mu_1)(X_2 - \mu_2) \cdots (X_n - \mu_n)]}{\sqrt[n]{E[X_1 - \mu_1]^n \times E[X_2 - \mu_2]^n \cdots E[X_n - \mu_n]^n}} \quad (8)$$

Correlation is just a specific case of *comrelation* where $n = 2$. Again, the *comrelation* $\zeta_{12\dots n}$ is in the range of $[-1, 1]$ according to the Holder inequality.

2.1 Risky valuation without collateralization

Recovery assumptions are important for pricing credit derivatives. If the reference entity under a CDS contract defaults, the best assumption, as pointed out by J. P. Morgan (1999), is that the recovered value equals the recovery rate times the face value plus accrued interest¹. In other words, the recovery of par value assumption is a better fit upon the default of the reference entity, whereas the recovery of market value assumption is a more suitable choice in the event of a counterparty default².

Proposition 1: The value of the CDS is given by

$$V(t) = \sum_{i=1}^m E \left[\left(\prod_{j=0}^{i-1} O(T_j, T_{j+1}) \right) X_i | \mathcal{F}_t \right] + \sum_{i=1}^m E \left[\left(\prod_{j=0}^{i-2} O(T_j, T_{j+1}) \right) \Omega(T_{i-1}, T_i) R(T_{i-1}, T_i) | \mathcal{F}_t \right] \quad (9a)$$

where $t = T_0$ and

$$O(T_j, T_{j+1}) = 1_{(V(T_{j+1}) + X_{j+1}) \geq 0} \phi_B(T_j, T_{j+1}) + 1_{(V(T_{j+1}) + X_{j+1}) < 0} \phi_A(T_j, T_{j+1}) \quad (9b)$$

¹ In the market, there is an average accrual premium assumption, i.e., the average accrued premium is half the full premium due to be paid at the end of the premium.

² Three different recovery models exist in the literature. The default payoff is either i) a fraction of par (Madan and Unal (1998)), ii) a fraction of an equivalent default-free bond (Jarrow and Turnbull (1995)), or iii) a fraction of market value (Duffie and Singleton (1999)).

$$\begin{aligned}
\phi_A(T_j, T_{j+1}) = & \{ p_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) + q_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \varphi_A(T_{j+1}) \\
& + p_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \bar{\varphi}_A(T_{j+1}) + q_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \varphi_{AB}(T_{j+1}) \\
& + p_C(T_j, T_{j+1}) \sigma_{AB}(T_j, T_{j+1}) (1 - \varphi_A(T_{j+1}) - \bar{\varphi}_A(T_{j+1}) + \varphi_{AB}(T_{j+1})) \\
& + \sigma_{AC}(T_j, T_{j+1}) [p_B(T_j, T_{j+1}) (1 - \varphi_A(T_{j+1})) + q_B(T_j, T_{j+1}) (\bar{\varphi}_A(T_{j+1}) - \varphi_{AB}(T_{j+1}))] \\
& + \sigma_{BC}(T_j, T_{j+1}) [p_A(T_j, T_{j+1}) (1 - \bar{\varphi}_A(T_{j+1})) + q_A(T_j, T_{j+1}) (\varphi_A(T_{j+1}) - \varphi_{AB}(T_{j+1}))] \\
& + \theta_{ABC}(T_j, T_{j+1}) (-1 + \bar{\varphi}_A(T_{j+1}) - \varphi_{AB}(T_{j+1}) + \varphi_A(T_{j+1})) \} D(T_j, T_{j+1})
\end{aligned} \tag{9c}$$

$$\begin{aligned}
\phi_B(T_j, T_{j+1}) = & \{ p_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) + q_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \bar{\varphi}_B(T_{j+1}) \\
& + p_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \varphi_B(T_{j+1}) + q_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) p_C(T_j, T_{j+1}) \varphi_{AB}(T_{j+1}) \\
& + p_C(T_j, T_{j+1}) \sigma_{AB}(T_j, T_{j+1}) (1 - \varphi_B(T_{j+1}) - \bar{\varphi}_B(T_{j+1}) + \varphi_{AB}(T_{j+1})) \\
& + \sigma_{AC}(T_j, T_{j+1}) [p_B(T_j, T_{j+1}) (1 - \bar{\varphi}_B(T_{j+1})) + q_B(T_j, T_{j+1}) (\varphi_B(T_{j+1}) - \varphi_{AB}(T_{j+1}))] \\
& + \sigma_{BC}(T_j, T_{j+1}) [p_A(T_j, T_{j+1}) (1 - \varphi_B(T_{j+1})) + q_A(T_j, T_{j+1}) (\bar{\varphi}_B(T_{j+1}) - \varphi_{AB}(T_{j+1}))] \\
& + \theta_{ABC}(T_j, T_{j+1}) (-1 + \bar{\varphi}_B(T_{j+1}) - \varphi_{AB}(T_{j+1}) + \varphi_B(T_{j+1})) \} D(T_j, T_{j+1})
\end{aligned} \tag{9d}$$

$$\begin{aligned}
\Omega(T_j, T_{j+1}) = & \{ p_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) q_C(T_j, T_{j+1}) + q_A(T_j, T_{j+1}) p_B(T_j, T_{j+1}) q_C(T_j, T_{j+1}) \bar{\varphi}_B(T_{j+1}) \\
& + p_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) q_C(T_j, T_{j+1}) \varphi_B(T_{j+1}) + q_A(T_j, T_{j+1}) q_B(T_j, T_{j+1}) q_C(T_j, T_{j+1}) \varphi_{AB}(T_{j+1}) \\
& + q_C(T_j, T_{j+1}) \sigma_{AB}(T_j, T_{j+1}) (1 - \varphi_B(T_{j+1}) - \bar{\varphi}_B(T_{j+1}) + \varphi_{AB}(T_{j+1})) \\
& - \sigma_{AC}(T_j, T_{j+1}) [p_B(T_j, T_{j+1}) (1 - \bar{\varphi}_B(T_{j+1})) + q_B(T_j, T_{j+1}) (\varphi_B(T_{j+1}) - \varphi_{AB}(T_{j+1}))] \\
& - \sigma_{BC}(T_j, T_{j+1}) [p_A(T_j, T_{j+1}) (1 - \varphi_B(T_{j+1})) + q_A(T_j, T_{j+1}) (\bar{\varphi}_B(T_{j+1}) - \varphi_{AB}(T_{j+1}))] \\
& + \theta_{ABC}(T_j, T_{j+1}) (1 - \bar{\varphi}_B(T_{j+1}) + \varphi_{AB}(T_{j+1}) - \varphi_B(T_{j+1})) \} D(T_j, T_{j+1})
\end{aligned} \tag{9e}$$

where $R(T_j, T_{j+1}) = (N(1 - \varphi_C(T_{j+1})) - \alpha(T_j, T_{j+1}))$, $\alpha(T_j, T_{j+1}) = sN\delta(T_S, T)/2$, and $X_i = -sN\delta(T_j, T_{j+1})$.

Proof: See the Appendix.

We may think of $O(t, T)$ as the risk-adjusted discount factor for the premium and $\Omega(t, T)$ as the risk-adjusted discount factor for the default payment. Proposition 1 says that the pricing process of a multiple-payment instrument has a backward nature since there is no way of knowing which risk-adjusted discounting rate should be used without knowledge of the future value. Only on the maturity date, the value of an instrument and the decision strategy are clear. Therefore, the evaluation must be done in a backward fashion, working from the final payment date towards the present. This type of valuation process is referred to as backward induction.

Corollary 1: *If counterparties A and B are default-free, the value of the CDS is given by*

$$\begin{aligned}
V(t) = & \sum_{i=1}^m E \left[\left(\prod_{j=0}^{i-1} O(T_j, T_{j+1}) \right) X_i | \mathcal{F}_t \right] + \sum_{i=1}^m E \left[\left(\prod_{j=0}^{i-2} O(T_j, T_{j+1}) \right) \Omega(T_{i-1}, T_i) R(T_{i-1}, T_i) | \mathcal{F}_t \right] \\
= & \sum_{i=1}^m E [D(t, T_i) p_C(t, T_i) X_i | \mathcal{F}_t] + \sum_{i=1}^m E [D(t, T_i) p_C(t, T_{i-1}) q_C(T_{i-1}, T_i) R(T_{i-1}, T_i) | \mathcal{F}_t]
\end{aligned} \tag{10}$$

where $O(T_{i-1}, T_i) = D(T_{i-1}, T_i) p_C(T_{i-1}, T_i)$; $\Omega(T_{i-1}, T_i) = D(T_{i-1}, T_i) q_C(T_{i-1}, T_i)$.

The proof of this corollary becomes straightforward according to Proposition 1 by setting $\rho_{kl}=0$,

$$\varphi_{AB} = 0, \varphi_{ABC} = 0, p_j = 1, q_j = 0, p_C(t, T_i) = \prod_{g=0}^{i-1} p(T_g, T_{g+1}), \text{ and } D(t, T_i) = \prod_{g=0}^{i-1} D(T_g, T_{g+1}).$$

If we further assume that the discount factor and the default probability of the reference entity are uncorrelated and the recovery rate φ_C is constant, we have

Corollary 2: Assume that i) counterparties A and B are default-free, ii) the discount factor and the default probability of the reference entity are uncorrelated; iii) the recovery rate φ_C is constant; the value of the CDS is given by

$$V(t) = \sum_{i=1}^m P(t, T_i) \bar{p}_C(t, T_{i-1}) \bar{q}_C(T_{i-1}, T_i) (N(1 - \varphi_C) - \alpha(T_{i-1}, T_i)) - \sum_{i=1}^m P(t, T_i) \bar{p}_C(t, T_i) sN\delta(T_{i-1}, T_i) \quad (11)$$

where $P(t, T_i) = E[D(t, T_i) | \mathcal{F}_t]$ denotes the bond price, $\bar{p}_C(t, T_i) = E[p_C(t, T_i) | \mathcal{F}_t]$, $\bar{q}_C(t, T_i) = 1 - \bar{p}_C(t, T_i)$, $\bar{p}(t, T_{i-1}) \bar{q}(T_{i-1}, T_i) = \bar{p}(t, T_{i-1}) - \bar{p}(t, T_i)$.

This corollary is easily proved according to Corollary 1 by setting $E[XY | \mathcal{F}_t] = E[X | \mathcal{F}_t] E[Y | \mathcal{F}_t]$ when X and Y are uncorrelated. Corollary 2 is the formula for pricing CDS in the market.

Our methodology can be extended to the cases where the number of parties $n \geq 4$. A generating function for the (probability) joint distribution (see details in Teugels (1990)) of n -variate Bernoulli can be expressed as

$$p^{(n)} = \begin{bmatrix} p_n & -1 \\ q_n & 1 \end{bmatrix} \otimes \begin{bmatrix} p_{n-1} & -1 \\ q_{n-1} & 1 \end{bmatrix} \otimes \dots \otimes \begin{bmatrix} p_1 & -1 \\ q_1 & 1 \end{bmatrix} \sigma^{(n)} \quad (12)$$

where $\otimes$ denotes the Kronecker product; $p^{(n)} = \{p_k^{(n)}\}$ and $\sigma^{(n)} = \{\sigma_k^{(n)}\}$ are vectors containing 2^n

components: $p_k^{(n)} = p_{k_1, k_2, \dots, k_n}$, $k = 1 + \sum_{i=1}^n k_i 2^{i-1}$, $k_i \in \{0, 1\}$; $\sigma_k^{(n)} = \sigma_{k_1, k_2, \dots, k_n} = E\left(\prod_{i=1}^n (Y_i - q_i)^{k_i}\right)$.

2.2 Risky valuation with collateralization

Collateralization is the most important and widely used technique in practice to mitigate credit risk. The posting of collateral is regulated by the Credit Support Annex (CSA) that specifies a variety of terms

including the threshold, the independent amount, and the minimum transfer amount (MTA), etc. The threshold is the unsecured credit exposure that a party is willing to bear. The minimum transfer amount is the smallest amount of collateral that can be transferred. The independent amount plays the same role as the initial margin (or haircuts).

The main role of collateral should be viewed as an improved recovery in the event of a counterparty default. According to Bankruptcy law, if there has been no default, the collateral is returned to the collateral giver by the collateral taker. If a default occurs, the collateral taker possesses the collateral. In other words, collateral does not affect the survival payment; instead, it takes effect on the default payment only.

According to the ISDA (2013), almost all CDSs are fully collateralized. Many people believe that full collateralization can eliminate counterparty risk completely for CDS.

We assume that a CDS is fully collateralized, i.e., the posting of collateral is equal to the amount of the current *MTM* value: $C(t) = V(t)$. For a discrete one-period (t, u) economy, there are several possible states at time u : i) A , B , and C survive with probability p_{000} . The instrument value is equal to the market value $V(u)$; ii) A and B survive, but C defaults with probability p_{001} . The instrument value is the default payment $R(u)$; iii) For the remaining cases, either or both counterparties A and B default. The instrument value is the future value of the collateral $V(t) / D(t, u)$ (Here we consider the time value of money only). The value of the collateralized instrument at time t is the discounted expectation of all the payoffs and is given by

$$\begin{aligned} V(t) &= E\{D(t, u)[p_{000}(t, u)V(u) + p_{001}(t, u)R(u) \\ &\quad + (p_{100}(t, u) + p_{010}(t, u) + p_{110}(t, u) + p_{101}(t, u) + p_{011}(t, u) + p_{111}(t, u))V(t) / D(t, u)] | \mathcal{F}_t\} \\ &= E\{[D(t, u)(p_{000}(t, u)V(u) + p_{001}(t, u)R(u)) + (1 - p_{000}(t, u) - p_{001}(t, u))V(t)] | \mathcal{F}_t\} \end{aligned} \quad (14a)$$

or

$$\begin{aligned} &E[(p_A(t, u)p_B(t, u) + \sigma_{AB}(t, u)) | \mathcal{F}_t]V(t) \\ &= E\{[D(t, u)(p_C(t, u)V(u) + q_C(t, u)R(u))(p_A(t, u)p_B(t, u) + \sigma_{AB}(t, u)) \\ &\quad + D(t, u)(V(u) - R(u))(p_B(t, u)\sigma_{AC}(t, u) + p_A(t, u)\sigma_{BC}(t, u) - \theta_{ABC}(t, u))] | \mathcal{F}_t\} \end{aligned} \quad (14b)$$

If we assume that $(p_A(t, u)p_B(t, u) + \sigma_{AB}(t, u))$ and $D(t, u)(p_C(t, u)V(u) + q_C(t, u)R(u))$ are uncorrelated, we have

$$V(t) = V^F(t) + \xi_{ABC}(t, u) / \psi_{AB}(t, u) \quad (15a)$$

where

$$V^F(t) = E\{D(t, u)[p_C(t, u)V(u) + q_C(t, u)R(u)] | \mathcal{F}_t\} \quad (15b)$$

$$\psi_{AB}(t, u) = E\{[p_A(t, u)p_B(t, u) + \sigma_{AB}(t, u)] | \mathcal{F}_t\} \quad (15c)$$

$$\xi_{ABC}(t, u) = E\{[D(t, u)(p_B(t, u)\sigma_{AC}(t, u) + p_A(t, u)\sigma_{BC}(t, u) - \theta_{ABC}(t, u))(V(u) - R(u))] | \mathcal{F}_t\} \quad (15d)$$

The first term $V^F(t)$ in equation (15) is the counterparty-risk-free value of the CDS and the second term is the exposure left over under full collateralization, which can be substantial.

Proposition 2: *If a CDS is fully collateralized, the risky value of the CDS is NOT equal to the counterparty-risk-free value, as shown in equation (15).*

Proposition 2 or equation (15) provides a theoretical explanation for the failure of full collateralization in the CDS market. It tells us that under full collateralization the risky value is in general not equal to the counterparty-risk-free value except in one of the following situations: i) the market value is equal to the default payment, i.e., $V(u) = R(u)$; ii) firms A , B , and C have independent credit risks, i.e., $\rho_{ij} = 0$ and $\varsigma_{ABC} = 0$; or iii) the following relationship holds $p_B\sigma_{AC} + p_A\sigma_{BC} = \theta_{ABC}$.

4 Conclusion

This article presents a new valuation framework for pricing financial instruments subject to credit risk. In particular, we focus on modeling default relationships.

We study the sensitivity of the price of a defaultable instrument to changes in the joint credit quality of the parties. For instance, our analysis shows that the effect of default dependence on CDS premia from large to small is the correlation between the protection seller and the reference entity, the comrelation, the

correlation between the protection buyer and the reference entity, and the correlation between the protection buyer and the protection seller.

Appendix

Proof of Proposition 1. Let $t = T_0$. On the first payment date T_1 , let $V(T_1)$ denote the market value of the CDS excluding the current cash flow X_1 . There are a total of eight ($2^3 = 8$) possible states shown in Table A_1.

Table A_1. Payoffs of a trilaterally defaultable CDS

This table shows all possible payoffs at time T_1 . In the case of $V(T_1) + X_1 \geq 0$ where $V(T_1)$ is the market value excluding the current cash flow X_1 , there are a total of eight ($2^3 = 8$) possible states: i) A , B , and C survive with probability p_{000} . The instrument value equals the market value: $V(T_1) + X_1$. ii) A defaults, but B and C survive with probability p_{100} . The instrument value is a fraction of the market value: $\bar{\varphi}_B(T_1)(V(T_1) + X_1)$ where $\bar{\varphi}_B$ represents the non-default recovery rate of party B ³. $\bar{\varphi}_B = 0$ represents the one-way settlement rule, while $\bar{\varphi}_B = 1$ represents the two-way settlement rule. iii) A and C survive, but B defaults with probability p_{010} . The instrument value is given by $\varphi_B(T_1)(V(T_1) + X_1)$ where φ_B represents the default recovery rate of defaulting party B . iv) A and B survive, but C defaults with probability p_{001} . The instrument value is the default payment: $R(T_0, T_1)$. v) A and B default, but C survives with probability p_{110} . The

³ There are two default settlement rules in the market. The *one-way payment rule* was specified by the early ISDA master agreement. The non-defaulting party is not obligated to compensate the defaulting party if the remaining market value of the instrument is positive for the defaulting party. The *two-way payment rule* is based on current ISDA documentation. The non-defaulting party will pay the full market value of the instrument to the defaulting party if the contract has positive value to the defaulting party.

instrument value is given by $\varphi_{AB}(T_1)(V(T_1) + X_1)$ where φ_{AB} denotes the joint recovery rate when both parties A and B default simultaneously. vi) A and C default, but B survives with probability p_{101} . The instrument value is a fraction of the default payment: $\bar{\varphi}_B(T)R(T_0, T_1)$. vii) B and C default, but A survives with probability p_{011} . The instrument value is given by $\varphi_B(T)R(T_0, T_1)$. viii) A , B , and C default with probability p_{111} . The instrument value is given by $\varphi_{AB}(T)R(T_0, T_1)$. A similar logic applies to the case of $V(T_1) + X_1 < 0$.

Status	Probability	Payoff if $V(T_1) + X_1 \geq 0$	Payoff if $V(T_1) + X_1 < 0$
$Y_A = 0, Y_B = 0, Y_C = 0$	p_{000}	$V(T_1) + X_1$	$V(T_1) + X_1$
$Y_A = 1, Y_B = 0, Y_C = 0$	p_{100}	$\bar{\varphi}_B(T_1)(V(T_1) + X_1)$	$\varphi_A(T_1)(V(T_1) + X_1)$
$Y_A = 0, Y_B = 1, Y_C = 0$	p_{010}	$\varphi_B(T_1)(V(T_1) + X_1)$	$\bar{\varphi}_A(T_1)(V(T_1) + X_1)$
$Y_A = 0, Y_B = 0, Y_C = 1$	p_{001}	$R(T_0, T_1)$	$R(T_0, T_1)$
$Y_A = 1, Y_B = 1, Y_C = 0$	p_{110}	$\varphi_{AB}(T_1)(V(T_1) + X_1)$	$\varphi_{AB}(T_1)(V(T_1) + X_1)$
$Y_A = 1, Y_B = 0, Y_C = 1$	p_{101}	$\bar{\varphi}_B(T)R(T_0, T_1)$	$\bar{\varphi}_B(T)R(T_0, T_1)$
$Y_A = 0, Y_B = 1, Y_C = 1$	p_{011}	$\varphi_B(T)R(T_0, T_1)$	$\varphi_B(T)R(T_0, T_1)$
$Y_A = 1, Y_B = 1, Y_C = 1$	p_{111}	$\varphi_{AB}(T)R(T_0, T_1)$	$\varphi_{AB}(T)R(T_0, T_1)$

The risky price is the discounted expectation of the payoffs and is given by

$$\begin{aligned}
V(t) = E \left\{ \left[\mathbf{1}_{(V(T_1) + X_1) \geq 0} \left((p_{000}(T_0, T_1) + p_{100}(T_0, T_1)\bar{\varphi}_B(T_1) + p_{010}(T_0, T_1)\varphi_B(T_1) + p_{110}(T_0, T_1)\varphi_{AB}(T_1))(V(T_1) + X_1) \right) \right. \right. \\
+ \mathbf{1}_{(V(T_1) + X_1) < 0} \left((p_{000}(T_0, T_1) + p_{100}(T_0, T_1)\varphi_A(T_1) + p_{010}(T_0, T_1)\bar{\varphi}_A(T_1) + p_{110}(T_0, T_1)\varphi_{AB}(T_1))(V(T_1) + X_1) \right) \\
+ (p_{001}(T_0, T_1) + p_{101}(T_0, T_1)\bar{\varphi}_B(T_1) + p_{011}(T_0, T_1)\varphi_B(T_1) + p_{111}(T_0, T_1)\varphi_{AB}(T_1))R(T_s, T_1) \Big] \mathcal{F}_t \Big\} \\
= E \left\{ [O(T_0, T_1)(V(T_1) + X_1) + \Omega(T_0, T_1)R(T_s, T_1)] \mathcal{F}_t \right\}
\end{aligned} \tag{A1a}$$

where

$$O(T_0, T_1) = \mathbf{1}_{(V(T_1) + X_1) \geq 0} \phi_B(T_0, T_1) + \mathbf{1}_{(V(T_1) + X_1) < 0} \phi_A(T_0, T_1) \tag{A1b}$$

$$\begin{aligned}
\phi_A(T_0, T_1) = & \{p_A(T_0, T_1)p_B(T_0, T_1)p_C(T_0, T_1) + q_A(T_0, T_1)p_B(T_0, T_1)p_C(T_0, T_1)\varphi_A(T_1) \\
& + p_A(T_0, T_1)q_B(T_0, T_1)p_C(T_0, T_1)\bar{\varphi}_A(T_1) + q_A(T_0, T_1)q_B(T_0, T_1)p_C(T_0, T_1)\varphi_{AB}(T_1) \\
& + p_C(T_0, T_1)\sigma_{AB}(T_0, T_1)(1 - \varphi_A(T_1) - \bar{\varphi}_A(T_1) + \varphi_{AB}(T_1)) \\
& + \sigma_{AC}(T_0, T_1)[p_B(T_0, T_1)(1 - \varphi_A(T_1)) + q_B(T_0, T_1)(\bar{\varphi}_A(T_1) - \varphi_{AB}(T_1))] \\
& + \sigma_{BC}(T_0, T_1)[p_A(T_0, T_1)(1 - \bar{\varphi}_A(T_1)) + q_A(T_0, T_1)(\varphi_A(T_1) - \varphi_{AB}(T_1))] \\
& + \theta_{ABC}(T_0, T_1)(-1 + \bar{\varphi}_A(T_1) - \varphi_{AB}(T_1) + \varphi_A(T_1))\}D(T_0, T_1)
\end{aligned} \tag{A1c}$$

$$\begin{aligned}
\phi_B(T_0, T_1) = & \{p_A(T_0, T_1)p_B(T_0, T_1)p_C(T_0, T_1) + q_A(T_0, T_1)p_B(T_0, T_1)p_C(T_0, T_1)\bar{\varphi}_B(T_1) \\
& + p_A(T_1, T_1)q_B(T_0, T_1)p_C(T_0, T_1)\varphi_B(T_1) + q_A(T_0, T_1)q_B(T_0, T_1)p_C(T_0, T_1)\varphi_{AB}(T_1) \\
& + p_C(T_0, T_1)\sigma_{AB}(T_0, T_1)(1 - \varphi_B(T_1) - \bar{\varphi}_B(T_1) + \varphi_{AB}(T_1)) \\
& + \sigma_{AC}(T_0, T_1)[p_B(T_0, T_1)(1 - \bar{\varphi}_B(T_1)) + q_B(T_0, T_1)(\varphi_B(T_1) - \varphi_{AB}(T_1))] \\
& + \sigma_{BC}(T_0, T_1)[p_A(T_0, T_1)(1 - \varphi_B(T_1)) + q_A(T_0, T_1)(\bar{\varphi}_B(T_1) - \varphi_{AB}(T_1))] \\
& + \theta_{ABC}(T_0, T_1)(-1 + \bar{\varphi}_B(T_1) - \varphi_{AB}(T_1) + \varphi_B(T_1))\}D(T_0, T_1)
\end{aligned} \tag{A1d}$$

$$\begin{aligned}
\Omega(T_0, T) = & \{p_A(T_0, T)p_B(T_0, T)q_C(T_0, T) + q_A(T_0, T)p_B(T_0, T)q_C(T_0, T)\bar{\varphi}_B(T_1) \\
& + p_A(T_0, T)q_B(T_0, T)q_C(T_0, T)\varphi_B(T_1) + q_A(T_0, T)q_B(T_0, T)q_C(T_0, T)\varphi_{AB}(T_1) \\
& + q_C(T_0, T)\sigma_{AB}(T_0, T)(1 - \varphi_B(T_1) - \bar{\varphi}_B(T_1) + \varphi_{AB}(T_1)) \\
& - \sigma_{AC}(T_0, T)[p_B(T_0, T)(1 - \bar{\varphi}_B(T_1)) + q_B(T_0, T)(\varphi_B(T_1) - \varphi_{AB}(T_1))] \\
& - \sigma_{BC}(T_0, T)[p_A(T_0, T)(1 - \varphi_B(T_1)) + q_A(T_0, T)(\bar{\varphi}_B(T_1) - \varphi_{AB}(T_1))] \\
& + \theta_{ABC}(T_0, T)(1 - \bar{\varphi}_B(T_1) + \varphi_{AB}(T_1) - \varphi_B(T_1))\}D(T_0, T)
\end{aligned} \tag{A1e}$$

Similarly, we have

$$V(T_1) = E\left\{[O(T_1, T_2)(X_2 + V(T_2)) + \Omega(T_1, T_2)R(T_1, T_2)]|\mathcal{F}_{T_1}\right\} \tag{A2}$$

Note that $O(T_0, T_1)$ is $\mathcal{F}_{T_1}$ -measurable. By definition, an $\mathcal{F}_{T_1}$ -measurable random variable is a random variable whose value is known at time T_1 . According to *taking out what is known* and tower properties of conditional expectation, we have

$$\begin{aligned}
V(t) = & E\{[O(T_0, T_1)(X_1 + V(T_1)) + \Omega(T_0, T_1)R(T_0, T_1)]|\mathcal{F}_t\} = E[O(T_0, T_1)X_1|\mathcal{F}_t] \\
& + E[\Omega(T_0, T_1)R(T_0, T_1)|\mathcal{F}_t] + E\{O(T_0, T_1)E\{[O(T_1, T_2)(X_2 + V(T_2)) + \Omega(T_1, T_2)R(T_1, T_2)]|\mathcal{F}_{T_1}\}|\mathcal{F}_t\} \\
= & \sum_{i=1}^2 E\left[\left(\prod_{j=0}^{i-1} O(T_j, T_{j+1})\right)X_i\right] + \sum_{i=1}^2 E\left[\left(\prod_{j=0}^{i-2} O(T_j, T_{j+1})\right)\Omega(T_{i-1}, T_i)R(T_{i-1}, T_i)\right] \\
& + E\left[\left(\prod_{j=0}^1 O(T_j, T_{j+1})\right)V(T_2)|\mathcal{F}_t\right]
\end{aligned} \tag{A3}$$

By recursively deriving from T_2 forward over T_m , where $V(T_m) = X_m$, we have

$$V(t) = \sum_{i=1}^m E\left[\left(\prod_{j=0}^{i-1} O(T_j, T_{j+1})\right)X_i|\mathcal{F}_t\right] + \sum_{i=1}^m E\left[\left(\prod_{j=0}^{i-2} O(T_j, T_{j+1})\right)\Omega(T_{i-1}, T_i)R(T_{i-1}, T_i)|\mathcal{F}_t\right] \tag{A4}$$

References

- Arora, Navneet, Priyank Gandhi, and Francis A. Longstaff (2009), “Counterparty credit risk and the credit default swap market,” Working paper, UCLA.
- Brigo, D., and Capponi, A., 2008, Bilateral counterparty risk valuation with stochastic dynamical models and application to Credit Default Swaps, Working paper.
- Das, S., D. Duffie, N. Kapadia, and L. Saita (2007), “Common failings: How corporate defaults are correlated,” *Journal of Finance*, 62: 93-117.
- Davis, M., and V. Lo (2001), “Infectious defaults,” *Quantitative Finance*, 1: 382-387.
- Crepey, S. (2015) “Bilateral counterparty risk under funding constraints – part II: CVA,” *Mathematical Finance*, 25(1), 23-50.
- Duffie, D., A. Eckner, G. Horel, and L. Saita (2009), “Frailty correlated default,” *Journal of Finance*, 64: 2089-2123.
- ECB (2009), “Credit default swaps and counterparty risk,” European central bank.
- FinPricing, Valuation service, <https://finpricing.com/lib/EqQuanto.html>
- Fuijii, M. and A. Takahashi (2012), “Collateralized CDS and default dependence – Implications for the central clearing,” *Journal of Credit Risk*, 8(3): 97-113.
- Gregory, Jon, 2009, Being two-faced over counterparty credit risk, *RISK*, 22, 86-90.

Hull, J. and A. White (2014), "Collateral and Credit Issues in Derivatives Pricing," *Journal of Credit Risk*, 10 (3), 3-28.

Jarrow, R., and S. Turnbull (1995), "Pricing derivatives on financial securities subject to credit risk," *Journal of Finance*, 50: 53-85.

Johannes, M. and S. Sundareshan (2007), "The impact of collateralization on swap rates," *Journal of Finance*, 62: 383-410.

J. P. Morgan (1999), "The J. P. Morgan guide to credit derivatives," Risk Publications.

Lando, D. and M. Nielsen (2010), "Correlation in corporate defaults: contagion or conditional independence?" *Journal of Financial Intermediation*, 19: 355-372.

Li, D. (2000), "On default correlation: A copula function approach," *Journal of Fixed Income*, 9: 43-54.

Longstaff, F., and E. Schwartz (2001): "Valuing American options by simulation: a simple least-squares approach," *The Review of Financial Studies*, 14 (1): 113-147.

Piterbarg, V. (2010), "Funding beyond discounting: collateral agreements and derivatives pricing," *Risk Magazine*, 2010 (2): 97-102.

Teugels, J. (1990), "Some representations of the multivariate Bernoulli and binomial distributions," *Journal of Multivariate Analysis*, 32: 256-268.

Zhang, G., and P. Jorion, (2007), “Good and bad credit contagion: Evidence from credit default swaps,”
Journal of Financial Economics 84: 860–883.