Branching Process Model applied to the $1/f$ Noise (II); Computer Simulations

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Abstract : Computer simulations are used to test how the branching process model is available to discuss the $1/f$ problem with the equations described in my previous paper, in which a particle may branch into several particles, be absorbed or be observed by a detector. In this work, it is assumed that exactly two particles are produced by a branching. The simulations demonstrate that the power spectrum of a series formed by time intervals between successive detections of particles is characterized by a $1/f$ distribution in a wide range of frequency (more than seven decades). It is also discussed that the $1/f$ fluctuations being familiar around us are strongly connected with a chain of causal relationships in a natural phenomenon, which reminds us of the law of Karma in Buddhism.

1. Introduction

Since the observation of the $1/f$ spectrum of shot noise by Johnson in 1925 [1], this kind of spectrum has been observed in a large number of phenomena. The aim of the present work is to apply the branching process model discussed in my old works [2,3] for generating a series whose spectrum is characterized by a $1/f$ distribution in a wide range of frequency and to point out that the $1/f$ phenomena are essentially connected with the observations of events on a complicated chain of causal relationships.

In the stochastic process such as the Markov process, an event in the process is really influenced by the event that happened just before. The time series composed of these events has usually a $1/f^2$ -like spectrum. In the present work, a medium having many particles is considered in which every single particle may branch into several particles and be absorbed, where the particle number at a moment is decided stochastically by that just before. If these particles are observed by a detector in the medium, the observed events may show another kind of stochastic behavior. This is the basic idea of my work [4] and is illustrated in Fig. 1.

Figure 1. Chains of the branching processes. The square, circle and black spot represent a particle immigrated randomly in the medium, absorption and detection of a particle, respectively. The path of a particle is given by a full line.

The particle numbers at time t_1 and t_2 in Fig. 1 are seven and eleven, respectively. A particle count may be correlated with another count through the branching paths as shown by a , b and c in the figure. The detection d has no correlation with a , b and c , because it is on another branching chain different from that of a , b and c . The length of the path between the counts has a statistical correlation with the physical time interval. The time interval, for example, between a and b is approximately equivalent to that between b and c , but the correlation between b and c may be far weaker than that between a and b , because the path between b and c is longer than that between a and b .

2. Detection probability

We focus only on the case where exactly two particles are produced by a branching reaction, and we consider a medium in which a particle may be subjected to absorption, branching reaction and detection with the rates λ_a , λ_m and λ_d , respectively. In my old work [2], only an absorption-type detector was considered where a particle is absorbed by detection, however, in my recent work [3], a non-absorption-type detector is also considered where the particle detection has no influence on the particle number as shown in Fig. 1. The statistics of the particles and particle-counts in the medium can be obtained in close forms, and their detailed mathematical expressions are obtained from the results in my work [3] in both cases where the detector is an absorption-type and is a non-absorption-type.

We consider the probability $P_k(m, n, t)$ that m counts have been recorded by a detector during the time interval $(0, t)$ and n particles are found in the medium at time $t > 0$ after k particles exist at $t = 0$ in the presence of random particle immigration with the rate S . It is rather complicated to obtain this probability for individual positive values of m , but when $m=0$, the probability $P_k(0, n, t)$ can be described in a close form as

$$P_k(0, n, t) = \sum_{i=0}^n K_k^{(0, n-i)} \cdot R_0^{(0, i)}, \quad (1)$$

where

$$K_k^{(0, n-j)} = \sum_{l=0}^{n-j} p(0, l, t) K_{k-1}^{(0, n-j-l)} \quad (2)$$

$$K_0^{(0, j)} = \delta_{j, 0},$$

and

$$p(0, l, t) = \begin{cases} \eta_0 \xi_0 V & (l = 0) \\ \frac{(\eta_0 - \xi_0)^2 e^{-\theta_0 t}}{(\eta_0 - \xi_0 e^{-\theta_0 t})^2} & (l = 1). \\ V \cdot p(0, l-1, t) & (l \geq 2) \end{cases} \quad (3)$$

Here

$$V = \frac{1 - e^{-\theta_0 t}}{\eta_0 - \xi_0 e^{-\theta_0 t}}. \quad (4)$$

The parameters θ_0 , η_0 and ξ_0 in Eqs. (3) and (4) are given differently depending on the type of detector of particles. When λ_c is the absorption rate of a particle in the medium except for that by a detector, the absorption rate λ_a defined before at the beginning of this section is given by

$$\lambda_a = \begin{cases} \lambda_c + \lambda_d & (\text{absorption type}) \\ \lambda_c & (\text{non-absorption type}) \end{cases} \quad (5)$$

Defining the multiplication rate μ and the detection rate ε of a particle, respectively, as

$$\mu = \frac{\lambda_m}{\lambda_a} \text{ and } \varepsilon = \frac{\lambda_d}{\lambda_a}, \quad (6)$$

which are the number of particles appearing in the medium by branching processes and the number of detected particles, respectively, during the mean lifetime of a particle absorbed in the medium ($1/\lambda_a$), the parameters θ_0 , η_0 and ξ_0 are expressed, respectively, as

$$\theta_0 = \begin{cases} \lambda_a \sqrt{(1 - \mu)^2 + 4\varepsilon\mu} & (\text{absorption-type}) \\ \lambda_a \sqrt{(1 + \varepsilon + \mu)^2 - 4\mu} & (\text{non-absorption type}) \end{cases}, \quad (7)$$

$$\eta_0 = \begin{cases} \frac{1}{2\mu} (1 + \mu + \sqrt{(1 - \mu)^2 + 4\varepsilon\mu}) & (\text{absorption-type}) \\ \frac{1}{2\mu} (1 + \varepsilon + \mu + \sqrt{(1 + \varepsilon + \mu)^2 - 4\mu}) & (\text{non-absorption type}) \end{cases} \quad (8)$$

and

$$\xi_0 = \begin{cases} \frac{1}{2\mu}(1 + \mu - \sqrt{(1 - \mu)^2 + 4\epsilon\mu}) & (\text{absorption - type}) \\ \frac{1}{2\mu}(1 + \epsilon + \mu - \sqrt{(1 + \epsilon + \mu)^2 - 4\mu}) & (\text{non - absorption type}) \end{cases}. \quad (9)$$

The other factor $R_0^{(0,i)}$ in Eq. (1) is given by

$$R_0^{(0,i)} = \begin{cases} \exp\left[(\xi_0 - 1)St + \frac{S}{\mu\lambda_a} \ln \frac{\eta_0 - \xi_0}{\eta_0 - \xi_0 e^{-\theta_0 t}}\right] & (i = 0) \\ \frac{S}{\mu\lambda_a} + i - 1 & \\ \frac{\mu\lambda_a}{i} \cdot V \cdot R_0^{(0,i-1)} & (i \geq 1) \end{cases}. \quad (10)$$

The probability that some counts during the time interval $(0, t)$ are recorded, i.e., $m > 0$, and n particles are found in the medium at $t > 0$ after we had k particles at $t = 0$ is given by

$$\sum_{m=1}^{\infty} P_k(m, n, t) = P_k(n, t) - P_k(0, n, t). \quad (11)$$

where $P_k(n, t)$ is the probability that n particles are found in the medium at time $t > 0$ after we had k particles at $t = 0$ and is given by

$$P_k(n, t) = \sum_{i=0}^n K_k^{(n-i)} Q_0^{(i)}. \quad (12)$$

Here

$$K_k^{(n-i)} = \sum_{l=0}^{n-i} p(l, t) K_{k-1}^{(n-i-l)} \quad (13)$$

$$K_0^{(i)} = \delta_{i,0}.$$

Defining the parameter α as

$$\alpha = \lambda_a - \lambda_m = (1 - \mu)\lambda_a, \quad (14)$$

the function $p(l, t)$ in Eq. (13) is described as

$$p(l, t) = \begin{cases} W_1 & (l = 0) \\ \left(\frac{1 - \mu}{1 - \mu e^{-\alpha t}}\right)^2 e^{-\alpha t} & (l = 1) \\ \mu W_1 \cdot p(l - 1, t) & (l \geq 2) \end{cases} \quad (15)$$

in case $\mu \neq 1$ (subcritical case), and

$$p(l, t) = \begin{cases} W_2 & (l = 0) \\ \left(\frac{W_2}{\lambda_a t}\right)^2 & (l = 1) \\ W_2 \cdot p(l - 1, t) & (l \geq 2) \end{cases} \quad (16)$$

in case $\mu=1$ (critical case). Here, the functions W_1 and W_2 are given, respectively, by

$$W_1 = \frac{1 - e^{-\alpha t}}{1 - \mu e^{-\alpha t}}, \quad (17)$$

and

$$W_2 = \frac{\lambda_a t}{\lambda_a t + 1}. \quad (18)$$

The other factor in Eq. (12) $Q_0^{(i)}$ is obtained as, when $\mu \neq 1$,

$$Q_0^{(i)} = \begin{cases} \exp \left[\frac{S}{\mu \lambda_a} \ln \frac{1-\mu}{1-\mu e^{-\alpha t}} \right] & (i = 0) \\ \frac{S}{\mu \lambda_a} + i - 1 & (i \geq 1) \end{cases} \mu W_1 \cdot Q_0^{(i-1)}, \quad (19)$$

and, when $\mu=1$,

$$Q_0^{(i)} = \begin{cases} \exp \left[-\frac{S}{\lambda_a} \ln (1 + \lambda_a t) \right] & (i = 0) \\ \frac{S}{\lambda_a} + i - 1 & (i \geq 1) \end{cases} W_2 Q_0^{(i-1)}. \quad (20)$$

When the left side in Eq. (11) is much smaller than $P_k(0, n, t)$ due to a very short time interval, the probability of recording more than two counts may be negligible and the following relation holds approximately.

$$P_k(1, n, t) \doteq P_k(n, t) - P_k(0, n, t). \quad (21)$$

3. Computer simulations

Whether particle detection has occurred or not in a short time interval was decided successively by using the Monte Carlo method with the probabilities given by Eqs. (1), (12) and (21) with a non-absorption-type detector, from which a series formed by the time intervals between two successive counts (detection series) was obtained. For avoiding the possibility that the particle number increases to infinity or dying out, the rate of random particle immigration was chosen to be $S = \alpha N_0$ when the multiplication rate $\mu < 1$ (subcritical case), considering the mean number of particles N_0 at $t \rightarrow \infty$ is S/α [2], and $S = 0$ when $\mu = 1$ (critical case). If $S > 0$ in a critical system, the particle number will increase with time and diverge eventually to infinity.

Figure 2. The fluctuations of (a) the particle series in case $\mu = 1$ and (b) the detection series in case $\mu = 1$ and $\varepsilon = 1$.

Too many particles existing lead to an unreasonably long time in processing on a computer and the number of particles should be set a limit. In the present simulations, the maximum number of particles was set at a limit of 5000, and the mean number of particles N_0 was fixed to 100 when $\mu < 1$. The short time interval t in Eq. (21) was fixed to 5×10^{-6} measured in the unit of $1/\lambda_a$ (mean a lifetime of an absorbed particle) in all the simulations with the branching process model in the present work. The fluctuations of a part of the detection series are shown in Fig. 2 in comparison with those of the series for the number of particles existing in the medium (particle series) simulated with the probability given by Eq. (12). The detection series has a much more intermittent property than the other.

Figure 3. The spectra of the detection series with a non-absorption-type detector in case $\mu = 1$ and $\varepsilon = 1$. The series sizes are (a) 2^{25} , (b) 2^{26} and (c) 2^{27} , respectively. The straight lines give the $1/f$ behavior.

The fast Fourier transformation (FFT) results for several sizes of count series in case $\mu = 1$ and the detection rate $\varepsilon = 1$ are shown in Fig. 3. When the series size is $2^{25} \cong 3.4 \times 10^7$, the spectrum behaves like $1/f$ over seven decades of frequency, here the frequency is related to particle counts and not to time. The spectrum for the series sizes with more than $2^{26} \cong 6.7 \times 10^7$ begins to deviate from the $1/f$ distribution in a low-frequency range. This deviation becomes more striking as the series size is longer and the spectral behavior converges to a finite value in a low-frequency range. In all the cases, the spectral behavior converges to a finite value in a high-frequency range. When $\varepsilon < 1$, it was examined in a similar way to the above that the spectrum deviates from the $1/f$ behavior in a low-frequency range starts at a shorter series size. The FFT results in case $\mu = 0.98$ and $\varepsilon = 1$ are shown in Fig. 4 where all the conditions except for μ are the same as those in case $\mu = 1$. When the series size is $2^{16} \cong 6.5 \times 10^4$, the spectrum behaves like $1/f$ for more than three decades of frequency, but that at the series size of $2^{17} \cong 1.3 \times 10^5$ begins to deviate from the $1/f$

distribution in a low-frequency range. In this case, the frequency range where the spectrum behaves like $1/f$ is much narrower than the case of $\mu = 1$.

The longest series sizes behaving like $1/f$ have been obtained at various values of μ with an

Figure 4. The spectra of the detection series with a non-absorption-type detector in case $\mu = 0.98$ and $\varepsilon = 1$. The series sizes are (a) 2^{16} and (b) 2^{17} , respectively. The straight lines give the $1/f$ behavior.

absorption-type detector as well as a non-absorption-type detector as shown in Fig. 5. The series size becomes sharply longer when $\mu > 0.98$. In cases $\mu = 0.90$ and 0.95 , the series sizes behaving like $1/f$ are slightly shorter when the particles are observed with an absorption-type detector compared to those with a non-absorption-type detector, but there is no clear difference between the results with these two types of detector at $\mu > 0.98$.

Figure 5. The longest series sizes behaving like $1/f$ at various values of μ . The circles and crosses are the results for the non-absorption-type and absorption-type detectors, respectively.

4. Discussions

From the simulations described above, it comes to light that a detection series formed by the time intervals between successive particle counts in a medium where a particle may branch into two particles has a spectrum characterized by a $1/f$ distribution and the frequency range behaving like $1/f$ depends strongly on the value of the multiplication rate μ . While this frequency range is very wide when $\mu = 1$, this range becomes sharply narrower with decreasing μ , as shown in Figs. 3, 4 and 5. The question, then, arises as to why the series size behaving like $1/f$ becomes so sharply shorter

with decreasing μ , i.e., the series size changes from about 3.3×10^7 to 6.5×10^4 while the value of μ decreases only 0.02, from 1 to 0.98, as seen in Fig. 5.

The spectra of particle series in cases $\mu = 1, 0.99, 0.98$ and 0.9 are given in Fig. 6. When $\mu = 1$, the spectrum behaves like $1/f^2$ at the series size of 2^{28} and starts to deviate from the $1/f^2$ distribution in a low-frequency range at the series size of 2^{29} . This situation is nearly kept even when $\mu = 0.99$, although the spectrum deviates slightly from the $1/f^2$ distribution at the series size of 2^{28} . On the other hand, when $\mu = 0.98$, this situation is realized only at a shorter series size, and at a much shorter series size when $\mu = 0.9$.

As can be seen in Fig. 6, the sizes of the particle series behaving like $1/f^2$ at $\mu = 0.98$ and 0.9 are, respectively, four times and sixteen times shorter compared with the case of $\mu = 0.99$. On the other hand, the particle series at $\mu = 0.99$ has a similar spectral characteristic to the case of $\mu = 1$, but the size of its detection series behaving like $1/f$ is more than 100 times shorter compared with the case of $\mu = 1$, as seen in Fig. 5.

Figure 6. The spectra of the particle series. The multiplication rate μ and the series size in the figure are, respectively, (a) $\mu = 1$ and 2^{28} , (b) $\mu = 1$ and 2^{29} , (c) $\mu = 0.99$ and 2^{28} , (d) $\mu = 0.99$ and 2^{29} , (e) $\mu = 0.98$ and 2^{26} , (f) $\mu = 0.98$ and 2^{27} , (g) $\mu = 0.9$ and 2^{24} , and (h) $\mu = 0.9$ and 2^{25} . The straight lines give the $1/f^2$ behavior.

Although the spectral characteristics of the particle series at $\mu = 0.99$ and $\mu = 1$ are quite similar to each other, why are their detection series sizes behaving like $1/f$ so different? As written before, the random immigration $S = 0$ when $\mu = 1$, and so all the particles in the medium are bred from a single original particle and are on the same branching chain as can be understood in Fig. 1. This means that every single particle in the medium has a correlation with all the other particles and,

therefore, every single count has a correlation with the other counts. On the other hand, as $S \neq 0$ when $\mu < 1$, some detection counts have no correlation with others like as the detection d in Fig. 1. The smaller the multiplication rate μ is, the more particles immigrate in the medium randomly in order to avoid the possibility of the particle number dying out and the more detection-counts having no correlation with others increase in number. These considerations suggest strongly that a correlation between particles on a single branching chain plays an important role in the spectral characteristics of a detection series behaving like $1/f$ in a wide range of frequencies.

The effect of the correlation between particles has been examined also by another kind of simulation, where it was assumed that the particle number n_0 at $t = 0$ will change, at a later time t , to another number n_t distributing normally around n_0 with a standard deviation $\sqrt{n_0}/N_{fix}$, where N_{fix} is a parameter describing the width of deviation. In the simulations, the maximum number of particles was also set at a limit of 5000 and the detection probability of an individual particle was set to $0.01/5000$ during the time interval $(0, t)$, which gives the maximum detection probability 0.01 when the particle number is 5000. In the simulations with the branching process model, we have a definite unit of time such as the mean lifetime of a particle absorbed in the medium, but there is no definite unit of time now, and so, the unit of time was supposed to be the time interval $(0, t)$. An example of the simulations is given in Fig. 7 in case $N_{fix} = 30$.

Figure 7. The spectra of (a) the particle series and (b) the detection series simulated with the model without assuming the branching processes. The straight lines give the $1/f^2$ and $1/f$ behavior in (a) and (b), respectively. When the series size is longer than 2^{24} in case of the particle series or 2^{16} in case of the detection series, the spectrum begins to deviate from the $1/f^2$ or $1/f$ distribution, respectively, in the low-frequency range.

In the present simulations, the particle number at a moment is really influenced just before, and the particle series has a spectrum characterized by a $1/f^2$ distribution. The correlation, however, between the particles in this model is uncertain. This comes out that the frequency ranges behaving

like $1/f^2$ and $1/f$ are not sufficiently wide in the spectra of the particle series and the detection series, respectively, as shown in Fig. 7.

The spatial distribution of particles and the interactions between them are not considered in the present work. These interactions make a correlation between particles across the chains of causal relationships originating from some immigrated particles as well as that between particles on a single chain, and therefore, for example, the detection d on a chain in Fig. 1 may correlate with the detections a , b and c , which are on another chain different from the above one, through the interactions between them. In this case, the expressions are similar to Eqs. (1) and (21) are probably much more complicated and their formalization may be extremely difficult. These interactions are, however, supposed to work on the chains of causal relationships in a similar way to the branching process, and a spectrum of a detection series generated considering the interactions between particles may also behave like $1/f$ in a wide range of frequency. When the particle density in the medium is not high, the interactions between particles are insignificant, and so it is expected to result in a similar conclusion to that obtained in this work.

5. Conclusions

The above discussions reveal that the relationship between particles is surely important to realize a $1/f$ spectrum in a wide range of frequencies. When $\mu = 1$, all the particles in the medium are related to the other particles on a single branching chain and, therefore, there is no particle independent of the other particles. This is the reason why the spectrum of a detection series behaves like $1/f$ in a wide range of frequencies.

This result obtained in the present research gives us an instructive suggestion about the $1/f$ fluctuations frequently observed in natural phenomena. The global environment is affected by sunlight, solar heat, the heat of the earth's interior, the rotation and revolution of the earth, and so on. The global situation at a moment is influenced by the conditions just before, and therefore its change with time is moderate as shown in Fig. 2(a). On the other hand, the local situation of the environment is so complicated and is different from place to place. A local situation at a place is an all-inclusive result of innumerable tiny events being correlative with each other and is influenced by many other events sometimes directly and sometimes indirectly, i.e., influenced directly by other events just beside and/or indirectly by other events far from this place through a complicated huge chain of causal relationships. In this way, there is a complicated chain of causal relationships between innumerable tiny events, like the particles on a branching chain as illustrated in Fig. 1. It is usually difficult to observe the whole of the events on the chain. We observe only a few of the events on a chain of causal relationships and, therefore, the relation between the observed events is complicated and difficult to understand clearly. Therefore, the observed values at a place influenced directly by the local natural conditions, for example, temperature, wind velocities, and so on at the place, may behave like $1/f$ in a wide range of frequency, because they are on a few huge chains of

causal relationships. On the other hand, it may be difficult to expect to observe fluctuations behaving like $1/f$ in a so wide range frequency, for example, in a traffic current on an expressway [5]. The cars in two successive interchanges have some relations with each other and they are on a chain of causal relationships, however, some cars in a longer current have no relation with others because they join the traffic line and make other chains such as the chain on which the detection d is in Fig. 1.

These above considerations suggest that our usual observation of part of the events on a chain of causal relationships may be a reason why we often find the $1/f$ phenomena around us. These situations remind us of the core idea of the law of Karma (dependent origination) in Buddhism.

In conclusion, the simulations with the branching process model reveal that a spectrum of a series formed by the time intervals between two successive particle counts by a detector behaves like $1/f$ in a wide range of frequency when the particles in the medium related to each other on a single branching chain. On the analogy of the above results, it is supposed further that the observation of part of events on a chain of causal relationships composing a natural phenomenon may come to a $1/f$ spectrum which is really familiar around us.

6. References

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