A Simple Proof of the Multinomial Theorem using Multivariate Calculus

Sumit Kumar Jha
IIIT-Hyderabad, India
Email: kumarjha.sumit@research.iiit.ac.in

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Let m, n be non-negative integers and $x_1, x_2, \dots, x_m$ be real numbers, then the multinomial theorem tells us that

$$f = (x_1 + x_2 + \dots + x_m)^n = \sum_{k_1+k_2+\dots+k_m=n} \frac{n!}{k_1!k_2!\dots k_m!} x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}. \quad (1)$$

Now we proceed with the proof. Applying the distributive law and collecting terms, we can see that

$$(x_1 + x_2 + \dots + x_m)^n = \sum_{k_1+\dots+k_m=n} C(n, k_1, \dots, k_m) x_1^{k_1} x_2^{k_2} \dots x_m^{k_m},$$

where $C(n, k_1, \dots, k_m)$ denotes the coefficient of $x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}$ ($\sum_{i=1}^m k_i = n$) in the expansion. Now, observe that

$$C(n, k_1, \dots, k_m) = \frac{1}{k_1!k_2!\dots k_m!} \left[\frac{\partial^n f}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_m^{k_m}} \right]_{x_1, x_2, \dots, x_m=0},$$

that is, in words, we first differentiate f exactly k_1 times with respect to x_1 , divide by $k_1!$, then differentiate further exactly k_2 times with respect to x_2 divide by $k_2!$, and so on until differentiating with respect to x_m exactly k_m times and divide by $k_m!$; and to finally make sure we get desired coefficient plug all $x_1, \dots, x_m$ to be 0.

But observe that

$$\begin{aligned} \frac{\partial^{k_1} f}{\partial x_1^{k_1}} &= (n)(n-1)\dots(n-k_1+1) \cdot (x_1 + \dots + x_m)^{n-k_1} \\ \frac{\partial^{k_1+k_2} f}{\partial x_2^{k_2} \partial x_1^{k_1}} &= (n)(n-1)\dots(n-(k_1+k_2)+1) \cdot (x_1 + \dots + x_m)^{n-k_1-k_2} \\ &\vdots \\ \frac{\partial^{k_1+\dots+k_m} f}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_m^{k_m}} &= n! \cdot (x_1 + \dots + x_m)^0 = n!. \end{aligned}$$

This completes the proof.