SEARCHING FOR $\{0,1\}$ VECTOR MAXIMIZING THE NUMBER OF POSITIVE DOT PRODUCTS WITH MATRIX ROWS

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ABSTRACT. This paper studies the following task: given an $n \times m$ $\{-1, +1\}$ matrix, search for an m -dimensional $\{0,1\}$ vector that maximizes the number of rows having positive dot product with it.

1. INTRODUCTION AND AN EXAMPLE

An example is given to clarify the description in the abstract. For the collection of vectors given by the rows of the following matrix:

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & +1 & -1 \\ -1 & +1 & +1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

the optimal choice of x is $[1, 1, 1, 0]^T$, which has positive dot products with the middle three rows. This is a search problem. To turn this into a decision problem, we can, instead, ask whether there exists an m -dimensional vector x that has positive dot product with k rows of the given matrix, where k is part of the input. A simplification of this would give a decision problem without k value given in the input. That is to ask whether there exists an m -dimensional vector x having positive dot product with every row of the matrix.

Now, we give the formal definition.

Definition 1.1. ZERO-ONE VECMAT DOTPROD

- **Input:** A collection of n m -dimensional vectors $v_1, \dots, v_n$ with every component in $\{-1, +1\}$
- **Output:** YES if there exists an m -dimensional vector x the dot product of which with every given vector is positive

We will prove that ZERO-ONE VECMAT DOTPROD is hard by a reduction from SET COVER, which is known to be hard since [1]. In the next section, we will give the sketch of the construction and then proceed to the correctness proof.

2. DETAILS OF THE CONSTRUCTION

We first give the sketch of the construction and then prove its correctness.

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2.1. Sketch of the construction. Suppose we are given a set cover instance consisting of number k and sets $S_1, S_2, \dots, S_m \subseteq \{1, 2, \dots, n\}$. The corresponding question is whether there exists a subset C of $\{1, 2, \dots, n\}$ with $|C| = k$ such that each S_i contains at least one element of C (or equivalently such that $S_i \cap C$ is non-empty for each i).

We use this input to construct an instance of your problem, that is, a set of vectors over $\{-1, 1\}$ of the same length. The length of these vectors is going to be $n + k + 1$, and there will be $m + k + 2$ of them. In particular, we will have vectors $a, b_1, b_2, \dots, b_{k+1}$, and $c_1, c_2, \dots, c_m$ as defined below.

Denote by $1(n)$ a vector of length n whose values are all 1s. Denote by $e_i(n)$ a vector of length n whose i th value is 1 and whose other values are -1 . Denote by s_j a vector of length n whose i th value is 1 if $i \in S_j$ and -1 otherwise.

Then

- $a = [1(k+1) ; -1(n)]$
- $b_j = [e_j(k+1) ; 1(n)]$ for $1 \leq j \leq k+1$ and
- $c_j = [-e_1(k+1) ; s_j]$ for $1 \leq j \leq m$

2.2. Proof of correctness. The proof of the correctness of the reduction given in subsection 2.1 is given in this subsection.

Proof. Let $x = [x' ; x'']$ be any vector of 0s and 1s expressed as two subvectors x' and x'' of lengths $k+1$ and n .

Suppose that the dot product of x with vectors $a, b_1, \dots, b_{k+1}$ is positive. Then consider the two vectors a and b_j (for some j). These two vectors are almost exactly negatives of each other. In particular, they both have 1 as the j th component but are otherwise negatives of each other (this is easy to verify). Then if x is a vector of 0s and 1s with a 0 in the j th component, the dot product of x with a is the negative of the dot product of x with b_j . In this case, these two dot products cannot both be positive. Therefore, x must have a 1 in the j th component.

The above logic can be applied for every j with $(1 \leq j \leq k+1)$. From this, we can conclude that if x has a positive dot product with vectors $a, b_1, \dots, b_{k+1}$, then $x' = 1(k+1)$.

Furthermore, we know that $x \cdot a > 0$, and

$$\begin{aligned} x \cdot a &= [x' ; x''] \cdot [1(k+1) ; -1(n)] \\ &= [1(k+1) ; x''] \cdot [1(k+1) ; -1(n)] \\ &= 1(k+1) \cdot 1(k+1) + x'' \cdot -1(n) \\ &= k+1 - |x''|_1 \end{aligned}$$

where $|x''|_1$ is the number of 1s in x'' , so $k+1 - |x''|_1 > 0$ and $|x''|_1 < k+1$.

Similarly, $x \cdot b_1 > 0$, and

$$\begin{aligned} x \cdot b_1 &= [x' ; x''] \cdot [e_1(k+1) ; 1(n)] \\ &= [1(k+1) ; x''] \cdot [e_1(k+1) ; 1(n)] \\ &= 1(k+1) \cdot e_1(k+1) + x'' \cdot 1(n) \\ &= -(k-1) + |x''|_1, \end{aligned}$$

so $-(k-1) + |x''|_1 > 0$ and $|x''|_1 > k-1$.

Together the above implies that $|x''|_1 = k$. To summarize, we have shown that if x has a positive dot product with vectors $a, b_1, \dots, b_{k+1}$, then $x' = 1(k+1)$ and $|x''|_1 = k$. In fact, it is easy to verify that the reverse is also true. In other

words, $x = [x' ; x'']$ has a positive dot product with vectors $a, b_1, \dots$, and b_{k+1} if and only if $x' = 1(k+1)$ and $|x''|_1 = k$.

From now on, consider only vectors x of the above form. Notice that there is a one to one correspondence between possible vectors x'' and subsets of $\{1, \dots, n\}$ of size k : in particular, x'' can be put into correspondence with set $X \subseteq \{1, \dots, n\}$ with $|X| = k$ such that $(x'')_i = 1$ iff $i \in X$.

Under this correspondence, we can consider the necessary and sufficient conditions for x to have positive dot product with vectors $c_1, \dots$, and, c_n .

For any j , we can expand $x \cdot c_j$ as

$$x \cdot c_j = [1(k+1) ; x''] \cdot [-e_1(k+1) ; s_j] = 1(k+1) \cdot -e_1(k+1) + x'' \cdot s_j = (k-1) + x'' \cdot s_j.$$

Notice that the value of $x'' \cdot s_j$ is exactly a sum of k components of s_j : in particular, $x'' \cdot s_j = \sum_{i \in X} (s_j)_i$. If these k components are all -1 s, then this sum is $-k$, and so $x \cdot c_j = (k-1) + x'' \cdot s_j = (k-1) - k = -1 < 0$. If these k components include at least one 1 , then this sum is at least $-1 \times (k-1) + 1 \times 1 = -(k-2)$ in which case $x \cdot c_j = (k-1) + x'' \cdot s_j \geq (k-1) - (k-2) = 1 > 0$. Thus, we see that $x \cdot c_j \geq 0$ for a specific j if and only if X includes at least one i such that $(s_j)_i = 1$.

Remember that $(s_j)_i = 1$ if and only if $i \in S_j$. Thus, $x \cdot c_j \geq 0$ for a specific j if and only if X includes at least one i such that $i \in S_j$, or equivalently if and only if $X \cap S_j$ is non-empty.

Then $x \cdot c_j \geq 0$ for every j if and only if for every j , $X \cap S_j$ is not empty. Notice that the latter condition is exactly equivalent to the condition that X is a solution to the input set cover instance.

In summary, we see that x is a vector of 0s and 1s with positive dot product with all the vectors produced by the reduction if and only if $x = [1(k+1) ; x'']$ with $|x''|_1 = k$ and x'' corresponds to a set $X \subseteq \{1, \dots, n\}$ with $|X| = k$ such that X is a solution to the input instance of set cover. In other words, a solution to the set cover instance exists if and only if a solution to the instance produced by the reduction exists. $\square$

3. CONCLUSION

As long as we study a mathematical conjecture, we should convey ourselves of having enough labouring hours on Prasolov and Sharygin maths books. Then, reading some articles on theory of computing like [2] is a good practice. Only after that, could we think of the ultimate final for all mathematics sciences.

REFERENCES

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